# TOPICS IN DIFFERENTIAL GEOMETRY: HOMEWORK 04

DUE MARCH 17

We have shown that if $M$ is *compact*, then $J_M(\gamma) = \frac{1}{2}\int_{S^1} |\gamma'(t)|^2 \, dt$ defined for $\gamma \in L_1^2(S^1, M)$ satisfies Condition C.

(1) If $M$ is non-compact, show that $J_M$ cannot satisfy Condition C.

Two continuous maps $\gamma_0, \gamma_1 : S^1 \to M$ are said to be (free[1]) homotopic if there exists a continuous $H : S^1 \times [0,1] \to M$ such that $H(\,\cdot\,, 0) = \gamma_0(\,\cdot\,)$ and $H(\,\cdot\,, 1) = \gamma_1(\,\cdot\,)$.

For the cylinder $S^1 \times \mathbb{R}^1$, $S^1 \times \{t\}$ are homotopic for all $t$. These circles are called the latitude circles. Suppose that $\Sigma$ is homeomorphic to the cylinder. Let

$$\mathscr{M}_L = \{\gamma \in L_1^2(S^1, \Sigma) : \gamma \text{ is homotopic to the latitude}\} .$$

In fact, $\mathscr{M}_L$ is a connected component of $L_1^2(S^1, \Sigma)$. For the following surfaces in $\mathbb{R}^3$, determine whether $J_M$ on $\mathscr{M}_L$ satisfies Condition C or not. Give your reason.

(2) Let $(u(t), v(t)) : (0, \infty) \to \mathbb{R}^2$ be a curve with the following properties.
- It always lies in the first quadrant, $u(t) > 0$ and $v(t) > 0$ for all $t$.
- When $t < 1$, $u(t) = t$ and $v(t) = 1/t^2$.
- When $t > 100$, $u(t) = t - \tanh(t)$ and $v(t) = \operatorname{sech}(t)$.

Let $M$ be the surface of revolution about the $u$-axis: $(u(t), v(t)\cos\theta, v(t)\sin\theta)$.

(3) The catenoid: $(t, \cosh(t)\cos\theta, \cosh(t)\sin\theta)$ where $t \in \mathbb{R}$.

[1] No base point condition is imposed.