TOPICS IN DIFFERENTIAL GEOMETRY: HOMEWORK 03

DUE MARCH 10

(1) For $2 \leq p < \infty$, consider $E = L^p(S^1, \mathbb{R}^N)$, which is the completion of $C^\infty(S^1, \mathbb{R}^N)$ with respect to the norm

$$||\phi||_{L^p} = \left(\int_{S^1} |\phi(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}} .$$

Let f be the function on E defined by

$$f(\phi) = \int_{S^1} (1 + |\phi(t)|^2)^{\frac{p}{2}} \, \mathrm{d}t .$$

(a) Show that f is well-defined, i.e. the integral converges.
(b) Prove that f is C^2.
(c) Is f smooth? Justify your answer.

(2) Let Σ be a compact, oriented surface with a Riemannian metric. For any $p > 2$, prove that the function

$$f : L^p_1(\Sigma, \mathbb{R}) \to \mathbb{R} \quad \text{defined by} \quad f(\phi) = ||\phi||^p_{L^p_1}$$

is C^2.

Remark. This can be used to show that $L^p_1(\Sigma, M)$ admits partitions of unity of class C^2.