TOPICS IN DIFFERENTIAL GEOMETRY: HOMEWORK 02

DUE MARCH 3

In this homework set, Σ is a compact, oriented surface with a volume form μ_Σ; M is a complete, Riemannian manifold. Consider $\mathscr{M} = \mathrm{Map}(\Sigma, M)$ for some class of maps from Σ to M. Here, we will not bother with the class of maps, and only do formal calculation.

(1) Let $f : M \to \mathbb{R}$ be a smooth function.
   (a) Construct a function $\mathfrak{f} : \mathscr{M} \to \mathbb{R}$. That is to say, any $\gamma \in \mathscr{M}$ means a map from Σ to M, and $\mathfrak{f}(\gamma)$ assigns a real number for γ.
   (b) For any tangent vector $X \in T_\gamma\mathscr{M}$, compute $(\mathrm{d}\mathfrak{f})|_\gamma(X)$. Note that for any $p \in \Sigma$, $X(p)$ in a vector in $T_{\gamma(p)}M$.

(2) Let α be a smooth 1-form on M.
   (a) Construct a 1-form $\mathcal{A}$ on $\mathscr{M}$. To be more precise, for any $\gamma \in \mathscr{M}$ and $X \in T_\gamma\mathscr{M}$, give the expression of $\mathcal{A}|_\gamma(X)$.
   (b) Compute $\mathrm{d}\mathcal{A}$: for any $\gamma \in \mathscr{M}$ and $X, Y \in T_\gamma\mathscr{M}$, work out $(\mathrm{d}\mathcal{A})|_\gamma(X, Y)$.

   A possible approach to (b). Choose a local coordinate $\{u^i\}_{i=1}^n$ for M^n. The 1-form α is locally $\alpha_i(u)\mathrm{d}u^i$.

   Consider those maps whose image lies in a proper subset of this chart. In this case, those mappings can be regarded as a subset of $\mathscr{E} = \mathrm{Map}(\Sigma, \mathbb{R}^n)$. A tangent vector field on $\mathscr{E}$ is a map from $\mathscr{E}$ to $\mathscr{E}$. Originally, X and Y are vectors in $\mathscr{E}$; extend them to maps, $\mathscr{X}$ and $\mathscr{Y}$, from $\mathscr{E}$ to $\mathscr{E}$ such that $\mathscr{X}(\gamma) = X$ and $\mathscr{Y}(\gamma) = Y$. Note that $(D\mathscr{Y})(\mathscr{X})$ at γ can be computed as follows: Consider the curve $\gamma_s = \gamma + sX$ in $\mathscr{E}$. The derivative of its image under the map $\mathscr{Y}$ at $s = 0$ is by definition $(D\mathscr{Y})(\mathscr{X})$ at γ. Namely,

$$(D\mathscr{Y})|_\gamma(X) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \mathscr{Y}(\gamma_s)\ .$$