TOPICS IN DIFFERENTIAL GEOMETRY: HOMEWORK 01

DUE FEBRUARY 24

Notation: $\mathbb{Z}_+ = \{j \in \mathbb{Z} : j \geq 0\}$.

(1) For $p = 2$, the Sobolev spaces are not only Banach spaces, but Hilbert spaces. One can also define them by using the Fourier transform.

A smooth function u defined on $\mathbb{R}^n$ is said to be of *Schwartz class* if for any $\alpha \in \mathbb{Z}_+^n$ and $k \in \mathbb{Z}_+$, there exists some $C_{\alpha,k} > 0$ such that

$$|D^\alpha u(x)| \leq C_{\alpha,k}(1 + |x|)^{-k} \quad \text{for any } x \in \mathbb{R}^n \ .$$

The operator D^α is $i^{-|\alpha|}\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$. Denote the space of these functions by $\mathscr{S}$. The Fourier transform

$$\hat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} u(x) \, \mathrm{d}x$$

defines an isomorphism $\hat{(\cdot)} : \mathscr{S} \to \mathscr{S}$. It obeys

$$u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \hat{u}(\xi) \, \mathrm{d}\xi \quad \text{(Fourier inversion formula)} \ ,$$

$$\widehat{D_x^\alpha u}(\xi) = \xi^\alpha \hat{u}(\xi) \ ,$$

$$\widehat{x^\alpha u}(\xi) = D_\xi^\alpha \hat{u}(\xi) \ ,$$

$$\int_{\mathbb{R}^n} u(x)\overline{v(x)} \, \mathrm{d}x = \int_{\mathbb{R}^n} \hat{u}(\xi)\overline{\hat{v}(\xi)} \, \mathrm{d}\xi \quad \text{(Plancherel's identity)} \ .$$

For any $k \in \mathbb{R}$, define the L^2_k norm by

$$||u||^2_{L^2_k} = \int_{\mathbb{R}^n} (1 + |\xi|)^{2k} |\hat{u}(\xi)|^2 \, \mathrm{d}\xi \ .$$

The L^2_k space is the completion of $\mathscr{S}$ in this norm.

(a) When $k \in \mathbb{Z}_+$, explain that the above norm is equivalent to the one defined in Moore's book.

(b) For any $\ell \in \mathbb{Z}$ with $\ell < k - \frac{n}{2}$, prove that there exists a constant $C_{k,\ell} > 0$ such that

$$||u||_{C^\ell} \leq C_{k,\ell} ||u||_{L^2_k}$$

for any $u \in \mathscr{S}$. (Hint: Try to use the Fourier inversion formula.)

(2) Let Σ be a compact[1], oriented surface. Show by an example that a function in $L^2_1(\Sigma, \mathbb{R})$ needs not to be continuous.

[1]Compactness is not important here.