DIFFERENTIAL GEOMETRY I: HOMEWORK 09

DUE NOVEMBER 25

(1) Suppose that $\gamma(t)$ is a geodesic, and let $V(t)$ is a Jacobi field along $\gamma(t)$. Assume for simplicity that $V(0) \neq 0$. Prove the $V(t)$ is indeed a variational field of a family of geodesics. (We have shown in class that the variational field of a family of geodesics must be a Jacobi field.)

Here is a possible construction. Let $\sigma(s) = \exp_{\gamma(0)}(sV(0))$ (The main property we need is that $\sigma(0) = \gamma(0)$ and $\sigma'(0) = V(0)$). Take a smooth vector field $W(s)$ along $\sigma(s)$ with $W(0) = V(0)$ and $(\nabla_{\sigma'(s)}W)|_{s=0} = (\nabla_{\gamma'(t)}V)|_{t=0}$. Consider $\gamma_s(t) = \exp_{\sigma(s)}(tW(s))$.

(2) Due to Homework 7, we define ∇ on the cotangent bundle by using the Levi-Civita connection of (M, g). One can further extend ∇ to type $(0, q)$-tensors by requiring

$$\nabla_X(\eta_1 \otimes \cdots \eta_q) = (\nabla_X \eta_1) \otimes \cdots \otimes \eta_q + \cdots + \eta_1 \otimes \cdots \otimes (\nabla_X \eta_q)$$

for any 1-forms $\eta_1, \ldots, \eta_q$.

(a) Remember that the Riemannian metric g itself is a $(0, 2)$-tensor. Show that $\nabla g = 0$. In fact, this is equivalent to that the Levi-Civita connection is a metric connection.

(b) Let S be a $(0, q)$-tensor. For any vector fields $X, V_1, \ldots, V_q$, check that

$$\begin{aligned} X(S(V_1, \ldots, V_q)) &= (\nabla_X S)(V_1, \ldots, V_q) \\ &\quad + S(\nabla_X V_1, V_2, \ldots, V_q) + \cdots + S(V_1, \ldots, V_{q-1}, \nabla_X V_q) \ . \end{aligned}$$

(3) Derive the second variational formula for the energy function at a geodesic: Let

$$\gamma(t, s) : [0, 1] \times (-\varepsilon, \varepsilon) \to M$$

be a smooth map, and $\gamma(t, 0)$ is a geodesic. Write $\gamma_s(t)$ of the s-fixed curve $\gamma(t, s)$. We have computed that for any $s \in (-\varepsilon, \varepsilon)$,

$$\frac{\mathrm{d}}{\mathrm{d}s} E[\gamma_s(t)] = -\int_0^1 \langle V, \nabla_T T \rangle \,\mathrm{d}t + \langle V, T \rangle\big|_{t=0}^1$$

where $T(t, s) = \frac{\partial}{\partial t}\gamma(t, s)$ and $V(t, s) = \frac{\partial}{\partial s}\gamma(t, s)$. Derive

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_{s=0} E[\gamma_s(t)] \ .$$

Your answer shall involve $\langle R(T, V)V, T \rangle$ along $\gamma_0(t)$. Note that $\nabla_T T$ vanishes along $\gamma_0(t)$.

(4) Suppose that $\gamma(t)$ is a geodesic, and let $V \in \Gamma(TM)$ be a Killing field.

(a) Show that $V|_{\gamma(t)}$ is a Jacobi field, by a conceptual argument.

(b) Show that $V|_{\gamma(t)}$ is a Jacobi field, by checking the Jacobi equation directly.

(c) Suppose that (M, g) is connected and complete (in the sense of Hopf–Rinow). If at some $p \in M$, $V|_p = 0$ and $(\nabla_X V)|_p = 0$ for any $p \in T_pM$, prove that V vanishes identically.