DIFFERENTIAL GEOMETRY I: HOMEWORK 08

DUE NOVEMBER 18

(1) Suppose that $\gamma(t)$ is a geodesic. Show that $|\gamma'(t)|$ is a constant.

(2) Let (M, g) be a Riemannian manifold. A self-diffeomorphism $\psi : M \to M$ is called an *isometry* if $\psi^* g = g$. Namely,

$$g(X, Y) = g((\mathrm{d}\psi)(X), (\mathrm{d}\psi)(Y))$$

for any $X, Y \in TM$.

Let $V \in \Gamma(TM)$ be a tangent vector field. It is called a *Killing field* (or an *infinitesimal isometry*) if the flow φ_t generated by V is an isometry for any t.

(a) On $(\mathbb{R}^n, g_{\mathrm{std}})$, a vector field V is equivalent to a map T_V from $\mathbb{R}^n$ to $\mathbb{R}^n$. Specifically, $T_V(x^1, \cdots, x^n) = (V^1(x), \cdots, V^n(x))$. The vector field V is called a *linear field* if the map T_V is a linear map. Show that a linear field V is a Killing field if and only if T_V is skew-symmetric.

(b) Let V be a Killing field on (M, g). Let U be the geodesic neighborhood, $\exp_p(B(0; \varepsilon))$, of p. Suppose that $V(p) = 0$, and $V(q) \neq 0$ for any $q \in U \setminus \{p\}$. Prove that V is tangent to the geodesic spheres[1] centered at p.

(c) Prove that V is a Killing field if and only if

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) = 0$$

for any $X, Y \in TM$. Note that it is equivalent to $g(\nabla_{\frac{\partial}{\partial x^i}} V, \frac{\partial}{\partial x^j}) + g(\nabla_{\frac{\partial}{\partial x^j}} V, \frac{\partial}{\partial x^i}) = 0$ for any i, j.

(d) Let V be a Killing field with $V(p) \neq 0$. Show that there exists a coordinate chart $(x^1, \cdots, x^n)$ near V such that in this coordinate system, the coefficient of the metric, $g_{ij}(x)$, is independent of x^1.

(3) $\boxed{\text{Bonus}}$ Remember that TM is itself a smooth manifold. Suppose that M carries a Riemannian metric g. Construct a Riemannian metric on TM by using g and its Levi-Civita connection (without using the partition of unity construction).

[1] $\exp_p(\partial B(0; \delta))$ for any $\delta \in (0, \varepsilon)$.