DIFFERENTIAL GEOMETRY I: HOMEWORK 07

DUE NOVEMBER 11

(1) Let (M^n, g) be an oriented, Riemannian manifold. Show that the volume form is well-defined.

(2) On a Riemannian manifold (M, g), show that

$$\begin{aligned} & X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ & \quad + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X) \end{aligned}$$

is $C^\infty(M;\mathbb{R})$-linear in X and Z.

(3) On a Riemannian manifold (M, g), denote by ∇ its Levi-Civita connection. Show that

$$\nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W$$

is $C^\infty(M;\mathbb{R})$-linear in U, V and W.

(4) The cotangent bundle is the space $T^*M = \amalg_{p\in M} T_p^*M$. A map α from M to T^*M with $\alpha(p) \in T_p^*M$ for all $p \in M$ is a 1-form.

Similarly, a connection on the cotangent bundle is a bilinear map

$$\begin{array}{rccc} \nabla : & \Gamma(TM) \otimes \Omega^1(M) & \to & \Omega^1(M) \\ & (X, \alpha) & \mapsto & \nabla_X \alpha \end{array}$$

which satisfies

- $\nabla_{f\cdot X}\alpha = f \cdot \nabla_X \alpha$,
- $\nabla_X(f\cdot\alpha) = f\cdot\nabla_X\alpha + X(f)\cdot\alpha$

for any $f \in C^\infty(M;\mathbb{R})$. Roughly speaking, it is a directional derivative for 1-forms.

(a) Suppose that there is a connection ∇ on the tangent bundle. Show that the condition

$$\mathrm{d}(\alpha(Y)) = (\nabla\alpha)(Y) + \alpha(\nabla Y)$$

uniquely defines a connection ∇ on the cotangent bundle.

(b) If $\nabla_{\partial_i}\partial_j = A_{ij}^k \partial_k$, work out $\nabla_{\partial_i}\mathrm{d}x^j$.

(5) Recall that an inner product, $\langle\ ,\ \rangle$, on a vector space E induces an isomorphism by

$$v \in E \mapsto \langle v, \cdot\rangle \in E^*\ .$$

On a Riemannian manifold (M, g), this gives a map from TM to T^*M which sends T_pM isomorphically to T_p^*M for all $p \in M$. It is not hard to check the map is smooth.

Thus, it induces a map from $\Gamma(TM)$ to $\Omega^1(M)$. The image of X under this map is usually denoted by $X^\flat$. By definition,

$$X^\flat(Y) = g(X, Y) \ .$$

The map admits an inverse $\Omega^1(M) \to \Gamma(TM)$, and is usually denoted by $\alpha^\sharp$. Namely,

$$g(\alpha^\sharp, Y) = \alpha(Y) \ .$$

Consider the Levi-Civita connection on TM, ∇. Prove that

$$(\nabla_X Y)^\flat = \nabla_X (Y^\flat)$$

where the connection on the right hand side is defined by (4) with the Levi-Civita connection.