DIFFERENTIAL GEOMETRY I: HOMEWORK 06

DUE OCTOBER 21

(1) Show that S^n is orientable. You may try to use (2.a) of Homework 5.

(2) Let M^n be a smooth manifold, and Σ^k be a submanifold of M, which is compact, oriented and boundaryless[1].

  (a) Show that the integration over Σ defines a homomorphism from $\mathrm{H}^k_{\mathrm{dR}}(M)$ to $\mathbb{R}$.

  (b) Show that $\mathrm{H}^{n-1}_{\mathrm{dR}}(\mathbb{R}^n \setminus \{0\})$ is non-trivial. You may try to use (1.c) of Homework 5.

(3) Recall that $\mathbb{RP}^n$ is the space of lines in $\mathbb{R}^{n+1}$, and can also be described as $S^n/\{\pm 1\}$.

  (a) Explain that $\mathbb{RP}^2$ is not orientable.

  (b) Show that $\mathbb{RP}^3$ is orientable.

(4) The main purpose of this exercise is to show that $\mathrm{H}^1_{\mathrm{dR}}(S^2)$ is trivial.

  (a) Suppose that α is a closed 1-form on $\mathbb{R}^2$. Construct a function on $\mathbb{R}^2$ so that its exterior derivative is α.

  Note that one may connect any (x, y) to the origin by $(0, 0) \to (x, 0) \to (x, y)$ or $(0, 0) \to (0, y) \to (x, y)$ (or other paths). Does the integration along these paths gives the same value?

  (b) Use the strategy of (a) to show that $\mathrm{H}^1_{\mathrm{dR}}(S^2)$ is trivial.

(5) [W, exercise 13 in ch.2] See another file.

[1] $\partial\S = \varnothing$.