DIFFERENTIAL GEOMETRY I: HOMEWORK 05

DUE OCTOBER 14

(1) Calculate the exterior derivative of the following differential forms.
   (a) $\mathrm{d}z + x\,\mathrm{d}y - y\,\mathrm{d}z$ on $\mathbb{R}^3$.
   (b) $\dfrac{x\,\mathrm{d}y - y\,\mathrm{d}x}{1 + x^2 + y^2}$ on $\mathbb{R}^2$.
   (c) $\dfrac{1}{|\mathbf{x}|^n}\displaystyle\sum_{j=1}^{n}(-1)^{j-1}x^j\,\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^j} \wedge \cdots \wedge \mathrm{d}x^n$ on $\mathbb{R}^n \backslash \{0\}$, where $\widehat{\cdot}$ means that the term is not there.

(2) Consider S^2 with the $\mathbf{u}$ and $\mathbf{v}$ coordinates described by (4) of Homework 2. For
   (a) $\dfrac{4}{(1 + (u^1)^2 + (u^2)^2)^2}\,\mathrm{d}u^1 \wedge \mathrm{d}u^2$ and
   (b) $\dfrac{u^1\,\mathrm{d}u^2 - u^2\,\mathrm{d}u^1}{(1 + (u^1)^2 + (u^2)^2)^2}$,

   work out its expression in $\mathbf{v}$ coordinate.

(3) Consider the 2-form on $\mathbb{R}^{2n}$ defined by
$$\omega = \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4 + \cdots + \mathrm{d}x^{2n-1} \wedge \mathrm{d}x^{2n} \ .$$
   Compute $\omega^n = \omega \wedge \omega \wedge \cdots \wedge \omega$.

(4) For any $\xi \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$, show that
$$\mathrm{d}(\xi \wedge \eta) = (\mathrm{d}\xi) \wedge \eta + (-1)^k \xi \wedge (\mathrm{d}\eta) \ .$$

(5) Verify Cartan's magic formula:
$$L_V \alpha = (\mathrm{d} \circ \iota_V + \iota_V \circ \mathrm{d})\alpha$$
   for $\alpha = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k$.

(6) For a 1-form α and two vector fields U and V, show that
$$(\mathrm{d}\alpha)(U, V) = U(\alpha(V)) - V(\alpha(U)) - \alpha([U, V]) \ .$$
   Cartan's magic formula may help here.