DIFFERENTIAL GEOMETRY I: HOMEWORK 04

DUE OCTOBER 7

(1) On $\mathbb{R}^3$, consider the vector fields

$$U = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \quad \text{and} \quad V = \frac{\partial}{\partial y} \ .$$

It is easy to see that the flow generated by V is $\psi_s(x_0, y_0, z_0) = (x_0, y_0 + s, z_0)$.

(a) Solve the flow φ_t generated by U.

(b) Compute $\mathrm{d}\varphi_{-t}(V(\varphi_t(x_0, y_0, z_0)))$, and $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \mathrm{d}\varphi_{-t}(V(\varphi_t(x_0, y_0, z_0)))$. Verify that the result coincides with $[U, V]$.

(c) For non-zero t and s, will $(\varphi_t \circ \psi_s)(x_0, y_0, z_0) = (\psi_s \circ \varphi_t)(x_0, y_0, z_0)$ for some (x_0, y_0, z_0)? Give your reason.

(2) Determine whether the following distribution is involutive. If it is, describe its integral surface.

(a) $\mathrm{span}\{\frac{\partial}{\partial x}, \cos x \frac{\partial}{\partial y} + \sin x \frac{\partial}{\partial z}\}$ in $\mathbb{R}^3$

(b) $\mathrm{span}\{\frac{\partial}{\partial x} - \frac{y}{x^2+y^2}\frac{\partial}{\partial z}, \frac{\partial}{\partial y} + \frac{x}{x^2+y^2}\frac{\partial}{\partial z}\}$ in $\mathbb{R}^3 \backslash \{z\text{-axis}\}$.

(3) (a) Consider the open unit ball, $B_1(0)$, in $\mathbb{R}^n$. For any two distinct points $p, q \in B_1(0)$, show that there is a diffeomorphism $\psi : B_1(0) \to B_1(0)$ which sends p to q.
Hint: Let σ be the line segment connecting p, q. Try to construct a vector field U which is equal to $q - p$ on a neighborhood of σ, and $\operatorname{supp} U \subset B_1(0)$.

(b) Let M be a compact, connected manifold. Let $\mathrm{Aut}(M)$ be the group[1] of self-diffeomorphisms of M. Prove that $\mathrm{Aut}(M)$ acts *transitively* on M. Namely, for any $p, q \in M$, there exists some $\varphi \in \mathrm{Aut}(M)$ such that $\varphi(p) = q$.

(4) Let φ_t be the one-parameter family of diffeomorphism generated by U. We have shown in class that

$$(\varphi_{-t})_*([V, W]) = [(\varphi_{-t})_*(V), (\varphi_{-t})_*(W)]$$

for any $t \in \mathbb{R}$. Use it to give another proof of the Jacobi identity.

[1] The binary operation is the composition.