DIFFERENTIAL GEOMETRY I: HOMEWORK 02

DUE SEPTEMBER 23

(1) Consider the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ where $(k, \ell) \in \mathbb{Z}^2$ acts on $\mathbb{R}^2$ by

$$(k, \ell) \cdot (x, y) = (x + k, y + \ell) \ .$$

Let $F : \mathbb{R} \to T^2$ be given by $f(t) = [(t, \sqrt{2}t)]$.

(a) Show that F is injective.

(b) Prove that F is not an embedding.

(2) [W, exercise 9 in ch.1] Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = x^3 + xy + y^3 + 1 \ .$$

For which points, $p = (0, 0)$, $p = (\frac{1}{3}, \frac{1}{3})$, $p = (\frac{-1}{3}, \frac{-1}{3})$, is $f^{-1}(f(p))$ an (embedded) submanifold in $\mathbb{R}^2$?

(3) Let $F : \mathbb{R}^3 \to \mathbb{R}^4$ be given by

$$F(x, y, z) = (x^2 - y^2, xy, yz, zx) \ .$$

Let S^2 be the unit 2-sphere in $\mathbb{R}^3$. Observe that $f = F|_{S^2}$ satisfies $f(x, y, z) = f(-x, -y, -z)$, so that it descends to a map

$$\tilde{f} : \mathbb{RP}^2 = S^2/\{\pm 1\} \to \mathbb{R}^4 \ .$$

Prove that $\tilde{f}$ is an embedding. [Hint: A bijective continuous map from a compact topological space to a Hausdorff topological space is a homeomorphism.]

(4) Consider the unit sphere S^2 in $\mathbb{R}^3$ with the stereographic projection:

$$\varphi_U^{-1} : \mathbb{R}^2 \to U = S^2 \setminus \{(0, 0, 1)\} \qquad \varphi_V^{-1} : \mathbb{R}^2 \to V = S^2 \setminus \{(0, 0, -1)\}$$

$$\mathbf{u} \mapsto \left(\frac{2\mathbf{u}}{1 + |\mathbf{u}|^2}, \frac{-1 + |\mathbf{u}|^2}{1 + |\mathbf{u}|^2}\right) \qquad \mathbf{v} \mapsto \left(\frac{2\mathbf{v}}{1 + |\mathbf{v}|^2}, \frac{1 - |\mathbf{v}|^2}{1 + |\mathbf{v}|^2}\right)$$

Write down the following two vector fields in terms of (V, φ_V):

(a) $u^1 \frac{\partial}{\partial u^1} + u^2 \frac{\partial}{\partial u^2}$;

(b) $-u^2 \frac{\partial}{\partial u^1} + u^1 \frac{\partial}{\partial u^2}$.

(5) Let M^m, N^n be smooth manifolds, and $F : M \to N$ be a smooth map. Suppose that there is a submanifold : L^ℓ in N^n.

(a) Prove that for any $q \in L$, one can find a coordinate chart of N at q, (V, φ) so that $\varphi(L \cap V) = (\mathbb{R}^\ell \times \{0\}) \cap \varphi(V)$.

(b) [Bonus] Suppose that for any $q \in L$ and $p \in M$ with $f(p) = q$,

$$T_q L + \mathrm{d}F|_p(T_p M) = T_q N \ .$$

Prove that $F^{-1}(Z)$ is a submanifold in M, and determine its dimension.