DIFFERENTIAL GEOMETRY I: HOMEWORK 11

DUE DECEMBER 16

(1) On $\mathbb{R}^n$ with the standard flat metric, show that if

$$\alpha = \sum_{i_1 < \cdots < i_k} \alpha_I \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \ ,$$

then

$$\Delta\alpha = - \sum_{i_1 < \cdots < i_k} \left(\sum_j \frac{\partial^2 \alpha_I}{\partial x^j \partial x^j} \right) \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \ .$$

Suppose that (M, g) is compact, oriented, and boundaryless; denote by $\mathrm{d}\mu_g$ the volume form.

(2) On a Riemannian manifold, the *divergence* of a vector field is the function

$$\mathrm{div}(V) = \sum_i \langle \nabla_{e_i} V, e_i \rangle \ ,$$

where $\{e_i\}$ is a (local) orthonormal frame. Show that $\int_M \mathrm{div}(V) \mathrm{d}\mu_g = 0$ for any vector field V.

(3) It is not hard to verify that for functions $C^\infty(M; \mathbb{R}) \ni f$,

$$\Delta f = -\,\mathrm{tr}(\nabla^2 f) = - \sum_i \left(\nabla_{e_i} \nabla_{e_i} f - \nabla_{\nabla_{e_i} e_i} f \right) \ .$$

Show that for any $\beta \in \Omega^k(M)$ and $f \in \mathcal{C}^\infty(M; \mathbb{R})$,

$$-\,\mathrm{tr}(\nabla^2 (f \cdot \beta)) = -f \cdot \mathrm{tr}(\nabla^2 \beta) - 2\nabla_{\nabla f} \beta + (\Delta f) \cdot \beta \ .$$

(4) The Green operator $G : \Omega^k(M) \to (\mathcal{H}^k)^\perp$ is defined by setting $G(\alpha)$ to be the unique solution of $\Delta\omega = \alpha - \pi_{\mathcal{H}}(\alpha)$ in $(\mathcal{H}^k)^\perp$, Prove that G takes bounded sequences into sequences with Cauchy subsequences.

(5) Consider Δ acting on $\Omega^k(M)$ for some fixed k. A real number λ is said to be an *eigenvalue* of Δ if there exists a non-trivial k-form α such that $\Delta\alpha = \lambda\alpha$, and α is called an *eigenform*, The eigenforms corresponding to a fixed λ constitute a subspace $E^k_\lambda(M)$ of $\Omega^k(M)$, which is called the *eigenspace* of the eigenvalue λ.
   (a) Prove that the eigenvalues are non-negative.
   (b) Prove that the eigenspaces must be finite dimensional.
   (c) Prove that the eigenvalues have no accumulation point.
   (d) Prove that eigenforms corresponding to distinct eigenvalues are L^2-orthogonal.

   For the existence and further properties, see [Warner, ex. 16 in ch. 6].