DIFFERENTIAL GEOMETRY I: HOMEWORK 10

DUE DECEMBER 9

(1) On a Riemannian manifold (M, g), let $\{e_i\}$ be a local, orthonormal frame for TM. Denote bt θ_i^j the coefficient 1-forms for the Levi-Civita connection:

$$\nabla e_i = \theta_i^j \otimes e_j \ .$$

Show that

$$\langle R(U,V)e_i, e_j\rangle = (\mathrm{d}\theta_i^j + \theta_k^j \wedge \theta_i^k)(U,V) \ .$$

(2) Consider the Poincaré disk model of the hyperbolic geometry:

$$g = \frac{4}{\left(1 - \sum_{j=1}^n (x^j)^2\right)^2} \sum_{j=1}^n \mathrm{d}x^j \otimes \mathrm{d}x^j$$

on $D = \{(x^1, \cdots, x^n) \in \mathbb{R}^n \mid \sum_{j=1}^n (x^j)^2 < 1\}$.

(a) Calculate its Levi-Civita connection.

(b) Describe its Riemann curvature tensor.

(3) Consider the metric

$$g = A(r)^2 \, \mathrm{d}r \otimes \mathrm{d}r + r^2 \, \mathrm{d}\phi \otimes \mathrm{d}\phi + r^2 \sin^2\phi \, \mathrm{d}\theta \otimes \mathrm{d}\theta$$

on $M = I \times \mathbf{S}^2$. Here, r is the coordinate on the interval I, and (ϕ, θ) is the spherical coordinate on $\mathbf{S}^2$.

(a) Calculate its Levi-Civita connection.

(b) Describe its Riemann curvature tensor.

(4) On $S^3 \subset \mathbb{R}^4$, consider the following 1-forms

$$\begin{aligned}
\sigma^1 &= -x^2\mathrm{d}x^1 + x^1\mathrm{d}x^2 - x^4\mathrm{d}x^3 + x^3\mathrm{d}x^4 \ , \\
\sigma^2 &= -x^3\mathrm{d}x^1 + x^4\mathrm{d}x^2 + x^1\mathrm{d}x^3 - x^2\mathrm{d}x^4 \ , \\
\sigma^3 &= -x^4\mathrm{d}x^1 - x^3\mathrm{d}x^2 + x^2\mathrm{d}x^3 + x^1\mathrm{d}x^2 \ .
\end{aligned}$$

By using $\sum_{j=1}^4 (x^j)^2 = 1$, one can show that $\underline{g} = \sum_{k=1}^3 \sigma^k \otimes \sigma^k$ is the standard round metric of radius 1 on S^3. The important relation you will need is

$$\mathrm{d}\sigma^i = 2\,\sigma^j \wedge \sigma^k \quad \text{for } (i,j,k) \text{ being cylic permutation of } (1,2,3) \ .$$

On $\{s > 1\} \times \mathbb{S}^3$, consider the following metric:

$$g = \frac{1}{4}\frac{s+1}{s-1}\mathrm{d}s \otimes \mathrm{d}s + 4\frac{s-1}{s+1}\sigma^3 \otimes \sigma^3 + (s^2-1)(\sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2) \ .$$

Show that the Ricci curvature of this metric vanishes identically.

Hint: You may do formal calculation first. Say, $\omega^0 = f(s)\mathrm{d}s$, $\omega^1 = a(s)\sigma^1$, $\omega^2 = b(s)\sigma^2$ and $\omega^3 = c(s)\sigma^3$ form an orthonormal trivializing sections for the contagent bundle. Then, $\mathrm{d}\omega^0 = 0$, and

$$\begin{aligned}
\mathrm{d}\omega^1 &= -\omega^1_0 \wedge \omega^0 - \omega^1_2 \wedge \omega^2 - \omega^1_3 \wedge \omega^3 \\
&= a'\,\mathrm{d}s \wedge \sigma^1 + 2a\,\sigma^2 \wedge \sigma^3 \\
&= \frac{a'}{f\,a}\,\omega^0 \wedge \omega^1 + \frac{2a}{bc}\,\omega^2 \wedge \omega^3 \ .
\end{aligned}$$

(5) Prove that in *three* dimensions, the whole Riemann curvature tensor is determined by the Ricci curvature tensor.

Hint: Do this at every point p. We may assume $g_{ij}(p) = \delta_{ij}$. Denote by $R_{j\,k\,j\,\ell}$ the components of the Riemann curvature tensor (at p):

$$R_{j\,k\,j\,\ell} = \langle R(\partial_j, \partial_\ell)\partial_k, \partial_j \rangle \ .$$

The Ricci curvature (at p) is

$$\mathrm{Ric}_{k\ell} = \sum_j R_{j\,k\,j\,\ell}$$

for $k, \ell \in \{1, 2, 3\}$. Since the Ricci curvature is symmetric in k, ℓ, there are six components. You are asked to show $\mathrm{Ric}_{k\ell}$ completely determines $R_{i\,j\,k\,\ell}$. Remember that the Riemann curvature tensor has some symmetries.

Remark: (4) shows that this cannot be true when $\dim > 3$: its Ricci curvature vanishes, but the Riemann curvature tensor has non-trivial components.