DIFFERENTIAL GEOMETRY I: HOMEWORK 01

DUE SEPTEMBER 16

(1) [W, exercise 18 in ch.1] Prove that a C^∞ function $f : \mathbb{R}^2 \to \mathbb{R}$ cannot be injective.

(2) [W, exercise 1 in ch.1] The d-sphere is the set

$$S^d = \{(a_1, \ldots, a_{d+1}) \in \mathbb{R}^{d+1} : \sum_{j=1}^{d+1} a_j^2 = 1\} \ .$$

Let $\mathbf{N} = (0, \ldots, 0, 1)$ and $\mathbf{S} = (0, \ldots, 0, -1)$. Construct a coordinate cover of S^d by using the stereographic projections $S^d \setminus \{\mathbf{N}\} \to \mathbb{R}^d$ and $S^d \setminus \{\mathbf{S}\} \to \mathbb{R}^d$, and explain that it gives S^d a structure of smooth manifold. To be more precise, work out the coordinate transition function to show that it is a diffeomorphism.

(3) (a) Demonstrate that the *unit tangent bundle* of S^2,

$$S^1(TS^2) = \{(p, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : p \in S^2, \ v \in T_pS^2 \text{ with } |v| = 1\} \ ,$$

is a smooth manifold by constructing a coordinate cover for it, and work out its coordinate transition.

(b) Identify (set-theoretically) $S^1(TS^2)$ with a matrix group.

(4) Let $\mathbf{Gr}(2, 4)$ be the space of 2-dimensional vector subspaces in $\mathbb{R}^4$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_4$ be the *standard basis* of $\mathbb{R}^4$.

(a) Let $U_{12} \subset \mathbf{Gr}(2, 4)$ be the 2-planes whose orthogonal projection onto the span$\{\mathbf{e}_1, \mathbf{e}_2\}$ is a linear isomorphism, i.e.

$$U_{12} = \{P \in \mathbf{Gr}(2, 4) : \pi_{12} : P \to \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\} \text{ is an isomorphism}\} \ .$$

Construct $\psi_{12} = (\varphi_{12})^{-1} : \mathbb{R}^4 \to U_{12}$ in a similar manner as that for the projective spaces.

(b) Similarly, construct U_{34} and $\psi_{34} = (\varphi_{34})^{-1}$. Work out

$$\varphi_{34} \circ \psi_{12} : \varphi_{12}(U_{12} \cap U_{34}) \to \varphi_{34}(U_{12} \cap U_{34}) \ .$$

(c) Similarly, construct U_{13} and $\psi_{13} = (\varphi_{13})^{-1}$. Work out

$$\varphi_{13} \circ \psi_{12} : \varphi_{12}(U_{12} \cap U_{13}) \to \varphi_{13}(U_{12} \cap U_{13}) \ .$$