GEOMETRY II: HOMEWORK 12

DUE JUNE 12

(1) Consider $\mathbb{CP}^n$ with the Fubini–Study metric. The group

$$\mathrm{U}(n+1) = \{A \in \mathrm{GL}(n+1;\mathbb{C}) : A^* = A^{-1}\}$$

has a standard action on $\mathbb{C}^{n+1}$.

(a) Show that the action induces an action on $\mathbb{CP}^n$.

(b) Show that $\mathrm{U}(n+1)$ acts on $\mathbb{CP}^n$

(i) transitively;

(ii) holomorphically;

(iii) isometrically.

(c) Describe the (Riemannian) sectional curvature of the Fubini–Study metric. (The Fubini–Study metric is unique up to scaling. Fix a factor for your convenience.)

(2) For a finite dimensional real vector space V, if there is an endomorphism J such that $J^2 = -\mathbf{I}$, show that V must be even dimensional.

(3) Let M be a (real) smooth manifold. Suppose that there is a $J \in \mathrm{End}(T_{\mathbb{R}}M)$ such that $J^2 = -\mathbf{I}$. For any two tangent vector fields X and Y, define

$$N_J(X,Y) = [X,Y] + J([JX,Y] + [X,JY]) - [JX,JY] \ .$$

(a) Show that N_J is $\mathcal{C}^\infty(M;\mathbb{R})$-linear in both arguments.

Hence, N_J is a tensor. It is called the *Nijenhuis tensor*. The Newlander–Nirenberg Theorem says that J comes from a complex manifold structure if and only if $N_J \equiv 0$.

bonus With J, define T to be the subbundle of $T_{\mathbb{C}}M = T_{\mathbb{R}}M \otimes \mathbb{C}$ whose local sections take the form $X - iJX$. Similarly, the local section of $\bar{T}$ takes the form $X + iJX$. Note that both T and $\bar{T}$ are ($\mathbb{C}$-)subbbundle of $T_{\mathbb{C}}M$. Denote the ($\mathbb{C}$-)dual by T^* and $\bar{T}^*$, respectively. It follows from $T^*_{\mathbb{C}}M = T^* \otimes \bar{T}^*$ that

$$\Lambda^2 T^*_{\mathbb{C}}M = (\Lambda^2 T^*) \oplus (T^* \otimes \bar{T}^*) \oplus (\Lambda^2 \bar{T}^*) \ .$$

Still use the notation $\Omega^{p,q}$ for the sections of $\Lambda^p T^* \otimes \Lambda^q \bar{T}^*$. Now, consider the operator

$$\begin{aligned} S : \ \Omega^{1,0} &\to \Omega^{0,2} \\ a &\mapsto \mathrm{pr}_{(0,2)}(\mathrm{d}a) \ . \end{aligned}$$

(b) Prove that S is a tensor (i.e. not a differential operator), and is equivalent to N_J.

When J comes from a complex manifold structure, one can compute S by using the *holomorphic coordinate*. It is clear that $S \equiv 0$ in this case.