GEOMETRY II: HOMEWORK 11

DUE JUNE 5

(1) Check that the Fubini–Study metric is indeed positive definite. Namely,

$$\omega = \frac{i}{(1+|\mathbf{z}|^2)^2}\left(\sum_{\mu=1}^{n}(1+|\mathbf{z}|^2-|z^\mu|^2)\mathrm{d}z^\mu\wedge\mathrm{d}\bar{z}^\mu-\sum_{\mu\neq\nu}\bar{z}^\mu z^\nu\mathrm{d}z^\mu\wedge\mathrm{d}\bar{z}^\nu\right)$$

is positive definite at any $\mathbf{z}\in\mathbb{C}^n$.

(2) Suppose that E is a holomorphic vector bundle over a complex manifold M, with a Hermitian metric. Suppose that $\{e_\mu\}_\mu$ gives a local, holomorphic trivialization of E, and denote $\langle e_\mu, e_\nu\rangle$ by $H_{\mu\bar{\nu}}$. Find a "simple" formula for the first Chern form of the Chern connection, $\frac{i}{2\pi}\,\mathrm{tr}(F^\nabla)$.

Hint: We already see that in the rank 1 case, the Hermitian bundle metric is given by an $\mathbb{R}_+$-valued function H, and the first Chern form is $-\frac{i}{2\pi}\partial\bar{\partial}\log H$.

(3) Construct a Kähler metric on $\mathrm{Gr}(2;4)$. Work out the Kähler form on a coordinate chart, and verify your answer is positive definite.

Hint: For $\mathrm{Gr}(k;n)$, there is a tautological $\mathbb{C}^k$-bundle over it, which is a subbundle of the trivial $\mathbb{C}^n$-bundle. You may try to imitate the construction of the Fubini–Study metric. Exercise (2) may be helpful.

(4) Suppose that $L\to M$ is a holomorphic line bundle with a Hermitian metric. Check that $\partial\bar{\partial}\log H$ is globally defined.

Hint: H is only locally defined. What do you know about the functions H for different holomorphic trivializations?