GEOMETRY II: HOMEWORK 10

DUE MAY 29

(1) By blowing up $\mathbb{C}^2$ at the origin, one gets (the total space of) L_{-1}. Denote by $f : L_{-1} \to \mathbb{C}^2$ the map introduced in lecture. The map f sends the zero section, $\mathbb{CP}^1$, to the origin, and is biholomorphic between $L_{-1}\backslash\{\text{zero section}\}$ and $\mathbb{C}^2\backslash\{0\}$.

Consider $S = \{(Z_0, Z_1) \in \mathbb{C}^2 : (Z_0)^2 = (Z_1)^3\}$. Note S is a complex submanifold of $\mathbb{C}^2$ except at 0. Denote by $\tilde{S}$ the closure of $f^{-1}(S\backslash\{0\})$. Show that $\tilde{S}$ is a complex submanifold in L_{-1}, and find out $\tilde{S} \cap \{\text{zero section}\}$.

(2) Consider the Hopf fibration

$$\begin{array}{rccc} \pi : & S^3 & \to & \mathbb{CP}^1 = S^2 \\ & (Z_0, Z_1) & \mapsto & [Z_0, Z_1] \ . \end{array}$$

Denote by $D_0 = \{[z, 1] : |z| \leq 1\}$ and $D_1 = \{[1, w] : |w| \leq 1\}$.

(a) Write down diffeomorphisms between $\pi^{-1}(D_j)$ and $D_j \times S^1$, for $j = 0, 1$.

(b) Clearly $S^3 = \pi^{-1}(D_0) \cup \pi^{-1}(D_1)$. Note that $\pi^{-1}(D_0) \cap \pi^{-1}(D_1)$ corresponds to the boundary of the two pieces. Work out the identification map between $\partial(D_0 \times S^1)$ and $\partial(D_1 \times S^1)$.

(3) Let M be a complex manifold of dimension n. One can form the holomorphic tangent bundle T, whose local section is spanned by $\{\frac{\partial}{\partial z_j}\}$. Consider the dual bundle T^*, which has $\{\mathrm{d}z_j\}$ as the local sections. The line bundle $\Lambda^n T^*$ is called the *canonical bundle* of M, and is usually denote by K_M.

Since the transition of K_M is holomorphic, itself is a complex manifold of dimension $n + 1$. Prove that the canonical bundle of K_M, K_{K_M}, is (holomorphically) trivial by constructing a nowhere zero holomorphic section.

(4) Let $M = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n (z_j)^2 = 1\}$. That is the sum of square of complex numbers, but not the norm squares.

(a) Show the M is a complex manifold of dimension $n - 1$.

(b) Show that M is homeomorphic to the total space of the tangent bundle of S^{n-1}. Hint: Write $z_j = x_j + iy_j$.

(c) Show that K_M is trivial by constructing a nowhere zero holomorphic section.