GEOMETRY II: HOMEWORK 09

DUE MAY 22

(1) Consider $\mathrm{Gr}(2,4) = \{2\text{-planes in } \mathbb{C}^4\}$, where 2-plane means 2-dimensional complex vector subspace. Denote by Z^j, $j = 1, \ldots, 4$ for the standard coordinate on $\mathbb{C}^4$.
   (a) For the 2-planes P which is surject to the Z^1Z^2-plane under the projection, explain how to construct a coordinate on this subset of $\mathrm{Gr}(2,4)$.
   (b) Do the same procedure for that with respect to the Z^3Z^4-plane. Determine the overlap region, and check that the coordinate transition is holomorphic.

(2) The Plücker embedding is the map defined by

$$\begin{array}{cccc} \iota : & \mathrm{Gr}(k,n) & \to & \mathbb{P}(\Lambda^k\mathbb{C}^n) = \mathbb{P}(\mathbb{C}^N) = \mathbb{CP}^{N-1} \\ & \mathrm{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_k\} & \mapsto & [\mathbb{C}\langle \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_n \rangle] \end{array}$$

where $N = \binom{n}{k}$. Focus on the case when $k = 2$ and $n = 4$. The map takes the following form:

$$\begin{array}{cccc} \iota : & \mathrm{Gr}(2,4) & \to & \mathbb{CP}^5 \\ & \mathrm{span}\left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \right\} & \mapsto & \begin{bmatrix} (u_1v_2 - u_2v_1, u_3v_4 - u_4v_3, u_1v_3 - u_3v_1, \\ u_2v_4 - u_4v_2, u_1v_4 - u_4v_1, u_2v_3 - u_3v_2) \end{bmatrix} \end{array}.$$

   (a) Check the map is holomorphic and injective.
   (b) Denote by $[(Z_0, Z_1, \ldots, Z_5)]$ the homogenous coordinate for $\mathbb{CP}^5$. Prove that $Z_0Z_1 - Z_2Z_3 + Z_4Z_5 = 0$ defines a (smooth) submanifold in $\mathbb{CP}^5$. Show that it is exactly the image of ι, namely, do the surjectivity part.

   (Not a) hint: For injectivity and surjectivity, it actually involves only linear algebra.

(3) Suppose that there is a group action of G on a (complex) manifold M proper, which is properly discontinuous and has no fixed point. Prove that the quotient space M/G is a manifold of the same dimension.

   Hint: Here is the key step. For any $p \in M$, find an open neighborhood W such that the points in W are not equivalent under the action of G. To start, there exists an open neighborhood of U_1 of p which is homeomorphic to the closed unit ball in the Euclidean space, and p is maps to the origin. Let $U_m \subset U_1$ corresponds to the closed ball of radius $1/m$.