GEOMETRY II: HOMEWORK 08

DUE MAY 15

The main purpose of this Homework set is to introduce the *global angular form*, which is the key tool in [S.-S. Chern, *A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds*, Ann. of Math. (2) **45** (1944), 747–752].

Suppose that E is a rank $2m$, oriented vector bundle. Its Euler form, $\mathrm{eu}(E)$, is a d-closed form. It is d-exact means that $\mathrm{eu}(E)$ is trivial in $\mathrm{H}^{2m}_{\mathrm{dR}}(M)$, which is not true in general. As a parenthetical remark, due to the Poincaré lemma, *locally* one can always express $\mathrm{Pf}(F^\nabla)$ as an exterior derivative of a $(2m-1)$-form, but doing so globally may be impossible.

Now, choose a bundle metric on E, and a metric connection ∇ for E. The *sphere bundle* of E is defined to be

$$\mathbf{S}(E) = \{v \in E : |v| = 1\} \ .$$

For any $p \in M$, $\mathbf{S}(E_p) = \pi^{-1}(p) \cap \mathbf{S}(E)$ is diffeomorphic to S^{2m-1}. The global angular form Θ is a $(2m-1)$-form on $\mathbf{S}(E)$ (not on M!) which has the following properties.

- $\Theta|_{\mathbf{S}(E_p)}$ is a volume form[1] on S^{2m-1}.
- $\mathrm{d}\Theta = (\mp)\,\mathrm{eu}(F^\nabla)$. To be more precise, the right hand side means the pull-back of the Euler form from M to $\mathbf{S}(E)$ by the projection π.

(1) When $m = 1$, choose local, oriented, orthonormal sections for E: e_1 and e_2. It gives a coordinate for the fibers by $\xi^1 e_1 + \xi^2 e_2$. The connection ∇ takes the form

$$\nabla \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix} = \mathrm{d} \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix} + \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix}$$

where a is a locally defined 1-form on M. Consider the restriction of

$$-\xi^2(\mathrm{d}\xi^1 + a\xi^2) + \xi^1(\mathrm{d}\xi^2 - a\xi^1) \qquad (\spadesuit)$$

on $\mathbf{S}(E)$.

(a) Check that, up to some constant multiple, $(\spadesuit)$ satisfies the desired two properties of the global angular form. Note that $\mathbf{S}(E)$ is given by the equation $(\xi^1)^2 + (\xi^2)^2 = 1$, and thus $\xi^1\mathrm{d}\xi^1 + \xi^2\mathrm{d}\xi^2 = 0$.

(b) Show that $(\spadesuit)$ is well-defined. Namely, it is invariant under SO(2)-bundle transitions, or equivalently, the choice of oriented, orthonormal sections.

[1] A volume form is a nowhere vanishing top-degree form.

(2) When $m = 2$, choose local, oriented, orthonormal sections for E: e_1, e_2, e_3 and e_4. It gives a coordinate for the fibers by $\sum_{j=1}^{4} \xi^j e_j$. Denote by a_i^j the coefficient 1-form of ∇, $\nabla e_i = a_i^j\, e_j$. Since ∇ is a metric connection, $a_i^j + a_j^i = 0$. It follows that

$$\nabla(\xi^i e_i) = (\mathrm{d}\xi^j) e_j + \xi^i\, a_i^j\, e_j \ .$$

In other words, the connection ∇ takes the form

$$\nabla \begin{bmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \\ \xi^4 \end{bmatrix} = \mathrm{d} \begin{bmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \\ \xi^4 \end{bmatrix} + \begin{bmatrix} 0 & a_2^1 & a_3^1 & a_4^1 \\ -a_2^1 & 0 & a_3^2 & a_4^2 \\ -a_3^1 & -a_3^2 & 0 & a_4^3 \\ -a_4^1 & -a_4^2 & -a_4^3 & 0 \end{bmatrix} \begin{bmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \\ \xi^4 \end{bmatrix} \ .$$

Consider the restriction of

$$\begin{aligned} &\xi^1(\mathrm{d}\xi^2 + a_i^2\xi^i) \wedge (\mathrm{d}\xi^3 + a_j^3\xi^j) \wedge (\mathrm{d}\xi^4 + a_k^4\xi^k) \\ &- \xi^2(\mathrm{d}\xi^1 + a_i^1\xi^i) \wedge (\mathrm{d}\xi^3 + a_j^3\xi^j) \wedge (\mathrm{d}\xi^4 + a_k^4\xi^k) \\ &\quad + \xi^3(\mathrm{d}\xi^1 + a_i^1\xi^i) \wedge (\mathrm{d}\xi^2 + a_j^2\xi^j) \wedge (\mathrm{d}\xi^4 + a_k^4\xi^k) \\ &\qquad - \xi^4(\mathrm{d}\xi^1 + a_i^1\xi^i) \wedge (\mathrm{d}\xi^2 + a_j^2\xi^j) \wedge (\mathrm{d}\xi^3 + a_k^3\xi^k) \end{aligned} \tag{♣}$$

on $\mathbf{S}(E)$. It is not hard to check that (♣) satisfies the first property of the global angular form.

(a) Use (♣) to construct a global angular form. You have to add some terms to (♣), which are invariant under SO(4)-bundle transitions, and which help to achieve the second property.
Hint: Something like $\varepsilon_{ijk}(\mathrm{d}\xi^i + a_\ell^i\xi^\ell) \wedge (F_a)_j^k$ could be useful, where F_a is the curvature.