# GEOMETRY II: HOMEWORK 07

DUE MAY 8

(1) Prove the Whitney formula.

(a) If $E_1$ and $E_2$ are complex vector bundles over $M$, then

$$\mathrm{c}_k(E_1 \oplus E_2) = \sum_{j=0}^{k} \mathrm{c}_j(E_1) \wedge \mathrm{c}_{k-j}(E_2)$$

where $\mathrm{c}_0(\,\cdot\,)$ is set to be 1.

(b) If $E_1$ and $E_2$ are real vector bundles over $M$, then

$$\mathrm{p}_k(E_1 \oplus E_2) = \sum_{j=0}^{k} \mathrm{p}_j(E_1) \wedge \mathrm{p}_{k-j}(E_2)\ .$$

(2) For any non-zero integer $n$, consider the complex line bundle, $L_n$, over $S^2$, as introduced in the lecture. Show that as the underlying $\mathbb{R}^2$-bundles, $L_n$ is isomorphic to $L_{-n}$.

(3) Write $S^4$ as $\mathbb{R}^4 \cup \mathbb{R}^4/\sim$ where the equivalence relation is given by

$$(y^0, y^1, y^2, y^3) = \frac{1}{|\mathbf{x}|^2}(x^0, -x^1, -x^2, -x^3)$$

on where $\mathbf{x} \neq 0$ and $\mathbf{y} \neq 0$. Declare $\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ to be the positive orientation. Denote by $U$ for the $\mathbb{R}^4$ with the $\mathbf{x}$ coordinate, and denote by $V$ for the $\mathbb{R}^4$ with the $\mathbf{y}$ coordinate.

Define a $\mathbb{C}^2$-bundle, $E$, over $S^4$ by the transition

$$g_{VU} = \begin{bmatrix} x^0 + ix^1 & x^2 + ix^3 \\ -x^2 + ix^3 & x^0 - ix^1 \end{bmatrix}\ .$$

(a) Check that the map which sends $(\mathbf{x}, (\xi_1, \xi_2)) \in U \times \mathbb{C}^2$ to $\mathbb{C}^4$ given by

$$\big(\xi_1, \xi_2, (x^0 + ix^1)\xi_1 + (x^2 + ix^3)\xi_2, (-x^2 + ix^3)\xi_1 + (x^0 - ix^1)\xi_2\big)$$

identifies $E$ as a (complex) subbundle of the trivial bundle $S^4 \times \mathbb{C}^4$.

(b) Define a connection $\nabla$ on $E$. Check that your expression does obey the transition of the connection.

(c) Evalaute $\int_{S^4} \mathrm{c}_2(E)$.