GEOMETRY II: HOMEWORK 06

DUE APRIL 24

(1) (a) Prove that the tautological line bundle, L, over $\mathbb{RP}^n$ is not a trivial bundle. Recall that $\mathbb{RP}^n = \mathbf{S}^n / \pm 1$.

(b) Recall that L is defined to be a subbundle in the trivial bundle, $\mathbb{RP}^n \times \mathbb{R}^{n+1}$. The trivial bundle comes with a bundle metric. Consider $L^\perp$. Show that $T\mathbb{RP}^n$ is isomorphic[1] to $\mathrm{Hom}(L, L^\perp) = L^\perp \otimes L^*$.

(2) Consider $\mathbb{CP}^1 = \{\text{complex lines in } \mathbb{C}^2\}$. It is the one-point compactification of $\mathbb{C}$, and is diffeomorphic to $\mathbb{S}^2$.

(a) Define analogously the tautological (complex) line bundle E over $\mathbb{CP}^1$.

(b) Recall that $\mathbb{CP}^1 = \dfrac{\mathbb{C} \cup \mathbb{C}}{z \sim w = z^{-1}}$. In terms of this coordinate cover, work out the transition function of the tautological bundle.

(3) Suppose that $\mathbb{R}^n \to E \xrightarrow{\pi} M$ is a vector bundle. Let

$$E \times_M E = \{(e_1, e_2) \in E \times E \mid \pi(e_1) = \pi(e_2)\} .$$

Namely, it associates $E_p \times E_p$ for every $p \in M$. Locally, $E|_{\mathcal{U}} = \mathcal{U} \times \mathbb{R}^n$, $E|_{\mathcal{U}} \times E|_{\mathcal{U}} = (\mathcal{U} \times \mathbb{R}^n) \times (\mathcal{U} \times \mathbb{R}^n)$, and $(E \times E)|_{\mathcal{U}} = \{((x, u), (x, v))\}$.

A bundle metric is a smooth map $\mathfrak{g} : E \times_M E \to \mathbb{R}$ which defines a inner product on E_p for every p.

(a) Prove that for any (real) vector bundle, the transition functions can be required to be orthogonal matrices, i.e.

$$g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{O}(k) \subset \mathrm{GL}(k; \mathbb{R}) .$$

Hints: Suppose that $\{s_j\}_{j=1}^k$ form local trivializing sections over some open set. Performing Gram-Schmidt process gives pointwise orthonormal sections over the same open set.

(b) Show that any real vector bundle is isomorphic (abstractly) to its dual bundle.

(4) For complex vector bundles, one can always construct a Hermitian bundle metric on them. Discuss what happens for (2.b) over $\mathbb{C}$. Will E isomorphic to E^* over $\mathbb{C}$?

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[1]Two vector bundles over M, $\pi_j : E_j \to M$, are said to be isomorphic if there is a smooth map $\Phi : E_1 \to E_2$ which commutes with π_j and which is a linear isomorphism over each $p \in M$.