GEOMETRY II: HOMEWORK 05

DUE APRIL 17

(1) Calculate the exterior derivative of the following differential forms.

(a) $\mathrm{d}z + x\,\mathrm{d}y - y\,\mathrm{d}z$ on $\mathbb{R}^3$.

(b) $\dfrac{x\,\mathrm{d}y - y\,\mathrm{d}x}{1 + x^2 + y^2}$ on $\mathbb{R}^2$.

(c) $\dfrac{1}{|\mathbf{x}|^n}\displaystyle\sum_{j=1}^{n}(-1)^{j-1}x^j\,\mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^j} \wedge \cdots \wedge \mathrm{d}x^n$ on $\mathbb{R}^n\backslash\{0\}$, where $\widehat{\cdot}$ means that the term is not there.

(2) Check that

$$\mathrm{d}(\omega \wedge \eta) = (\mathrm{d}\omega) \wedge \eta + (-1)^k \omega \wedge (\mathrm{d}\eta)$$

for any $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$.

(3) For any 1-form α and vector fields U, V, prove that

$$\mathrm{d}\alpha(U, V) = U(\alpha(V)) - V(\alpha(U)) - \alpha([U, V])\ .$$

Hint: This formula is local in nature. Due to $\mathbb{R}$-linearity, it suffices to show it for $\alpha = f(\mathbf{x})\,\mathrm{d}x^1$.

(4) Consider the following two parametrization for $\mathbb{S}^2$:

$$F_+(x^1, x^2) = \frac{1}{1 + (x^1)^2 + (x^2)^2}\left(2x^1, 2x^2, 1 - (x^1)^2 - (x^2)^2\right)\ ,$$
$$F_-(y^1, y^2) = \frac{1}{1 + (y^1)^2 + (y^2)^2}\left(2y^1, -2y^2, -1 + (y^1)^2 + (y^2)^2\right)\ .$$

For the following differentials forms, find their expression in terms of the $\mathbf{y}$-coordinate.

(a) $\dfrac{4}{(1 + (x^1)^2 + (x^2)^2)^2}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$.

(b) $\dfrac{x^1\,\mathrm{d}x^2 - x^2\,\mathrm{d}x^1}{(1 + (x^1)^2 + (x^2)^2)^2}$.

(5) Let $\alpha = P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$ is a smooth 1-form on $\mathbb{R}^3$. Suppose that S is a regular surface with boundary. Still denote by α the restriction of α on S. Check that the Stokes theorem

$$\iint_S \mathrm{d}\alpha = \int_{\partial S} \alpha$$

gives the usual Stokes theorem in vector calculus.

(6) Consider the 2-dimensional real projective space, $\mathbb{RP}^2 = S^2/\pm 1$. That is to say, it is the quotient of S^2 by the equivalence relation on antipodal points. Show that $\mathbb{RP}^2$ is not orientable.

Hint: There is a projection map $\pi : S^2 \to \mathbb{RP}^2$. If there is a nowhere vanishing 2-form μ on $\mathbb{RP}^2$, $\pi^*\mu$ would be a nowhere vanishing 2-form on S^2.