GEOMETRY II: HOMEWORK 04

DUE APRIL 10

(1) Let $\Sigma \subset \mathbb{R}^N$ be an n-dimensional submanifold, with the induced metric. Consider the restriction of the coordinate functions $\{w^\mu\}_{\mu=1}^N$. Show that

$$\sum_{\mu=1}^{N} |\nabla^\Sigma w^\mu|^2 = n \ .$$

Hint: $\nabla^{\mathbb{R}^N} w^\mu$ is the standard basis vector for $\mathbb{R}^N$, e_μ.

(2) Let $\Omega \subset \mathbb{R}^n$ be a compact, connected region. For simplicity, assume $\partial\Omega$ is a smooth, $(n-1)$-dimensional submanifold. Denote by $\mathbf{n}$ the unit outer normal vector field of $\partial\Omega$. Suppose that u is the solution to the following Neumann problem:

$$\begin{cases} \Delta u = c_0 & \text{on } \Omega \ , \\ \frac{\partial u}{\partial \mathbf{n}} = 1 & \text{on } \partial\Omega \ . \end{cases}$$

Here, $\Delta u = \sum_{j=1}^n \partial_j^2 u$ is the usual Laplacian on $\mathbb{R}^n$. The constant c_0 is determined by the Green's identity.

$$c_0 \operatorname{Vol}(\Omega) = \int_\Omega \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} = \operatorname{Vol}(\partial\Omega) \ .$$

Consider the usual gradient of u: $\nabla u = (\partial_1 u, \cdots, \partial_n u)$. Define the *lower contact set* of u by

$$\Gamma_u = \{x \in \Omega : u(y) - u(x) \geq \langle \nabla u(x), y - x \rangle \ \text{ for any } y \in \Omega\} \ .$$

It is the set of points x such that the tangent hyperplane to the graph u at x lies below u in all of Ω.

(a) Regard ∇u as a map from Ω to $\mathbb{R}^n$. Prove that $B_1 \subset (\nabla u)(\Gamma_u)$, where B_1 is the open unit ball in $\mathbb{R}^n$.

(b) The map ∇u is $(\partial_1 u, \cdots, \partial_n u)$, and thus its derivative is the Hessian matrix of u. It follows from part (a) that

$$\operatorname{Vol}(B_1) \leq \int_{\Gamma_u} \det(D(\nabla u)) \, \mathrm{d}x \ .$$

Use these to show the isoperimetric inequality

$$\operatorname{Vol}(\Omega)^{n-1} \leq \frac{1}{n^n \operatorname{Vol}(B_1)} \operatorname{Vol}(\partial\Omega)^n \ .$$

Remark. $\operatorname{Vol}(\partial B_1) = n \operatorname{Vol}(B_1)$.

(3) (continued from (4) of HW3) Consider the Clifford torus in S^3:

$$\Sigma = \{\frac{1}{\sqrt{2}}(\cos\alpha, \sin\alpha, \cos\beta, \sin\beta)\} \ .$$

Note that the unit normal ν of Σ in S^3 is

$$\nu = \frac{1}{\sqrt{2}}(\cos\alpha, \sin\alpha, -\cos\beta, -\sin\beta) \ .$$

Suppose that there is a one-parameter family of surface in S^3: Σ_t with

$$\Sigma_0 = \Sigma \quad , \quad \left.\frac{\partial \Sigma_t}{\partial t}\right|_{t=0} = f\nu \quad \left(\text{and} \quad \left.\frac{\partial^2 \Sigma_t}{\partial t^2}\right|_{t=0} \,/\!/\, \nu\right)$$

for some $f \in C^\infty(\Sigma)$. Show that

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \mathrm{Vol}(\Sigma_t) = \int_\Sigma \left(|\nabla f|^2 - 4|f|^2\right) \mathrm{dvol} \ .$$

Note. By taking constant functions, Σ is not stable in S^3. In fact, minimal submanifolds in the sphere cannot be stable. You saw this phenomenon for geodesics.