GEOMETRY II: HOMEWORK 03

DUE MARCH 27

(1) Choose a coordinate cover for a manifold M^n. A 2-tensor[1] consists of the following data:
- a vector valued function $S_{ij}(x)$ for $i, j \in \{1, \ldots, n\}$ on each coordinate chart;
- on the overlapping region, $S_{ij}(x) = \tilde{S}_{k\ell}(y(x)) \frac{\partial y^k}{\partial x^i} \frac{\partial y^\ell}{\partial x^j}$.

It is often written as $S_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j = \tilde{S}_{k\ell}\, \mathrm{d}y^k \otimes \mathrm{d}y^\ell$. The Riemannian metric and the second fundamental form are both 2-tensor.

With a metric on M, it induces a metric for 2-tensors by

$$\langle S_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j, T_{k\ell}\, \mathrm{d}x^k \otimes \mathrm{d}x^\ell \rangle = g^{ik}\, g^{j\ell}\, S_{ij}\, T_{k\ell} \ .$$

Show that ∇ defined in the first week is compatible with the metric. That is to say,

$$\frac{\partial}{\partial x^i} \langle S, T \rangle = \langle \nabla_{\partial_i} S, T \rangle + \langle S, \nabla_{\partial_i} T \rangle \ .$$

Remark. $\nabla S = \mathrm{d}x^i \otimes \nabla_{\partial_i} S$ is a 3-tensor, and one can define the norm/inner product analogously.

(2) Let $(\bar{M}, \bar{g})$ be a Riemannian manifold, and $M \subset \bar{M}$ be a submanifold. It is also a Riemannian manifold with $g = \bar{g}|_M$. Let f be a smooth function on $\bar{M}$. Check that $\nabla^M f$ is the orthogonal projection of $\nabla^{\bar{M}} f$ onto TM.

(3) Let (Σ, g) be a Riemannian manifold. For $f \in C^\infty(\Sigma)$, $\mathrm{d}f = \frac{\partial f}{\partial x^i} \mathrm{d}x^i$, $\nabla f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$, and $|\nabla f|^2 = |\mathrm{d}f|^2 = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j}$. Show that

$$\frac{1}{2} \Delta f^2 = f \Delta f + |\nabla f|^2 \ .$$

(4) Let S^3 be the unit sphere in $\mathbb{R}^4$, with the induced metric.
  (a) Check that the Clifford torus, $\{\frac{1}{\sqrt{2}}(\cos\alpha, \sin\alpha, \cos\beta, \sin\beta)\}$, is a minimal surface in S^3. Note that the Gauss curvature of the Clifford torus is zero.
  (b) Suppose that Σ is a minimal surface in S^3. Prove that $\Delta w^i = -2w^i$ for the coordinate functions $\{w^i\}_{i=1}^4$ of $\mathbb{R}^4$.

Minimal surface in S^3 is defined similarly, and is also the critical state of the volume functional. Suppose that $X(x^1, x^2)$ is a local parametrization for the surface Σ, which is an $\mathbb{R}^4$-valued function. The orthogonal complement of $T_p\Sigma$ in T_pS^3 is one dimensional. Choose a unit normal ν of Σ in S^3. The second fundamental form (of

[1] It is usually called tensor type of $(0, 2)$ in most textbooks.

Σ in S^3) is given by

$$h_{ij} = \langle \frac{\partial^2 X}{\partial x^i \partial x^j}, \nu \rangle \ .$$

The mean curvature is still defined to be $H = g^{ij} h_{ij}$. Here is a hint:

$$\frac{\partial^2 X}{\partial x^i \partial x^j} = \Gamma^k_{ij} \frac{\partial X}{\partial x^k} + h_{ij}\, \nu - g_{ij}\, X.$$

I leave it to you to think about this equation.