GEOMETRY II: HOMEWORK 02

DUE MARCH 20

(1) (a) Suppose the Σ is a regular surface in $\mathbb{R}^3$ without umbilical points[1]. Prove that Σ is a minimal surface if and only the Gauss map $N : \Sigma \to S^2$ satisfies

$$\langle \mathrm{d}N(u), \mathrm{d}N(v)\rangle = \lambda_p \langle u, v\rangle$$

for any $p \in \Sigma$ and $u, v \in T_p\Sigma$, where λ_p is a scalar only depends on p. Remember that $\mathrm{d}N : T_p\Sigma \to T_{N(p)}S^2$.

(b) Note that S^2 admits an isothermal coordinate system. There are many ways to construct it, for instance, the stereographic projection, or the geodesic polar coordinate.

Suppose that Σ is a minimal surface, and $K_p \neq 0$ at some $p \in \Sigma$. Prove that one can use the "inverse" of the Gauss map to construct a (local) isothermal coordinate on a neighborhood of p.

(2) The Enneper surface is given by

$$\left(u(1 - \frac{u^2}{3} + v^2), -v(1 - \frac{v^2}{3} + u^2), u^2 - v^2 \right)$$

for $(u, v) \in \mathbb{R}^2$.

(a) Check that it is minimal.

(b) Calculate its total Gauss curvature, $\iint K \mathrm{d}\sigma$.

(3) For a function $f : \mathbb{R}^n \to \mathbb{R}$, the induced metric on Γ_f has coefficients

$$g_{ij} = \delta_{ij} + \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} \ .$$

(a) Calculate g^{ij}.

(b) Show that Γ_f is minimal is also equivalent to

$$g^{ij}\frac{\partial^2 f}{\partial x^i \partial x^j} = 0 \ .$$

(4) For a hypersurface Σ in $\mathbb{R}^{n+1}$ with parametrization $X(x^1, \cdots, x^n)$,

$$\frac{\partial^2 X}{\partial x^i \partial x^j} = \Gamma^p_{ij}\frac{\partial X}{\partial x^p} + h_{ij}\,\nu$$

where ν is the unit normal vector field. Derive the Gauss equation

$$R^p_{\ kij} = g^{pq}h_{iq}h_{jk} - g^{pg}h_{ik}h_{jq} \ .$$

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[1] A point $p \in \Sigma$ is called a umbilical point if the two principal curvatures coincide, i.e. A is proportional to g at p.