GEOMETRY II: HOMEWORK 01

DUE MARCH 13

(1) Let $X(x^1, x^2)$ be a parametrization of a regular surface in $\mathbb{R}^3$. Define Γ_{ij}^k to be the coefficient of $(\frac{\partial^2 X}{\partial x^i \partial x^j})^T$ in $\frac{\partial X}{\partial x^k}$. Namely,

$$(\frac{\partial^2 X}{\partial x^i \partial x^j})^T = \Gamma_{ij}^k \frac{\partial X}{\partial x^k} \ .$$

Show that

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\ell} \left(\frac{\partial g_{i\ell}}{\partial x^j} + \frac{\partial g_{j\ell}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell} \right) \ .$$

(2) Show that $\nabla g = 0$; namely, $\nabla_{\frac{\partial}{\partial x^k}} (g_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j) = g_{ij;k} \mathrm{d}x^i \otimes \mathrm{d}x^j = 0$.

(3) Check that

$$\frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} (V^i \sqrt{\det g}) = \frac{\partial V^i}{\partial x^i} + \Gamma_{ik}^i V^k \ .$$

(4) Derive the minimal surface equation. For $f : U \subset \mathbb{R}^2 \to \mathbb{R}^1$, its graph $\Gamma_f = \{(x^1, x^2, f(x^1, x^2)) \in \mathbb{R}^3 : (x^1, x^2) \in U\}$ is a minimal surface if and only if f satisfies

$$\sum_i \frac{\partial}{\partial x^i} \left(\frac{\frac{\partial f}{\partial x^i}}{\sqrt{1 + |\nabla f|^2}} \right) = 0 \ .$$

Here, $\nabla f = (\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2})$ is the usual gradient on $\mathbb{R}^2$.