§I. Hirzebruch – Riemann – Roch

For simplicity, consider $L \to M$ a holomorphic line bundle
(the theory works for higher rank as well)

$\Omega^{0,q}(L) = \{$ smooth section of $\Lambda^q \overline{T}^* \otimes L\}$

$L|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{C}$ biholomorphically $\longrightarrow$ $\mathbb{C}^1$-valued

over $\mathcal{U}$, elements in $\Omega^{0,q}$ are spanned by $\varphi \cdot s$, $\varphi \in \Omega^{0,q}(\mathcal{U})$

$\bar\partial(\varphi\cdot s) = (\bar\partial\varphi)\cdot s + (-1)^q \varphi \wedge \bar\partial s$

is well-defined

$\bar\partial^2(\varphi\cdot s) = \bar\partial^2\varphi + (-1)^{q+1}\bar\partial\varphi\wedge\bar\partial s + (-1)^q\bar\partial\varphi\wedge\bar\partial s + \varphi\wedge\bar\partial^2\varphi$
$= 0$

$\rightsquigarrow \quad 0 \to \Omega^0(L) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \Omega^{0,q}(L) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \Omega^{0,n}(L) \to 0$

$n = \dim_{\mathbb{C}} M$

$$H^q(L) = \frac{\ker\bar\partial \text{ on } \Omega^{0,q}}{\operatorname{im}\bar\partial}$$

<u>theorem</u>

$$\sum_{q=0}^{n} (-1)^q \dim H^q(L) = \int_M \underbrace{e^{c_1(L)}\, Td(T)}_{\text{integrate the deg}=2n \text{ component}}$$

$$e^{c_1(L)} = 1 + c_1(L) + \frac{1}{2}(c_1(L))^2 + \frac{1}{6}(c_1(L))^3 + \cdots$$

$$Td(T) = 1 + \frac{1}{2}c_1(T) + \frac{1}{12}\left((c_1(T))^2 + c_2(T)\right) + \cdots$$

not going to explain how to construct it.

Namely, the Euler characteristic of this L-valued $\bar\partial$-complex is topological.

<u>remark</u> When $n=1$, M is a Riemann surface

deg 2 part of $e^{c_1(L)}\cdot Td(T) = c_1(L) + \frac{1}{2}c_1(T)$

$\int$ ——— $= \deg(L) + 1 - g$

which is the classical Riemann – Roch

<u>note</u> The most common application: If $H^{q\geq 1}(L) = 0$
then $\dim H^0(L)$ is given by a topological number

But $H^0(L) = \{$(global) holomorphic section of $L\}$.

§II. Kodaira Bochner formula

1° Suppose (M,ω) is Kähler. Endow L a Hermitian bundle metric. They induce a L^2-norm on $\Omega^{0,q}(L)$

$$\Rightarrow H^{0,q}(L) \cong \ker(\bar\partial^*\bar\partial + \bar\partial^*\bar\partial)$$

<u>remark</u> The adjoint operator depends on the choice of metric

2° Locally. $\|s\|^2 = e^\rho |s|^2$

$\Rightarrow$ Chern connection $\nabla s = \partial s + (\partial\rho)\, s$

$$F = d(\partial\rho) = \bar\partial\partial\rho = -\partial\bar\partial\rho = -\partial_\nu\bar\partial_\mu\rho\, dz^\nu \wedge d\bar z^\mu$$

$$c_1(L) = \frac{i}{2\pi} F = \frac{-i}{2\pi}\partial_\nu\bar\partial_\mu\rho\, dz^\nu\wedge d\bar z^\mu$$

3° $\nabla^{0,1}: \Gamma(\Lambda^q \bar T^* \otimes L) \longrightarrow \Gamma(\underbrace{\bar T^* \otimes \Lambda^q \bar T^*}_{\text{not alternating yet}} \otimes L)$

<u>goal</u> Compare $\bar\partial^*\bar\partial + \bar\partial\bar\partial^*$ with $(\nabla^{0,1})^*\nabla^{0,1}$

Let us do the $q=1$ case for simplicity

$$\varphi = \varphi_{\bar\alpha}\, d\bar z^\alpha$$

i) $\bar\partial\varphi = \bar\partial_\beta \varphi_{\bar\alpha}\, d\bar z^\beta \wedge d\bar z^\alpha = \frac{1}{2}(\bar\partial_\alpha \varphi_{\bar\beta} - \bar\partial_\beta\varphi_{\bar\alpha})\, d\bar z^\alpha\wedge d\bar z^\beta$

$\psi = \frac{1}{2}\psi_{\bar\mu\bar\nu}\, d\bar z^\mu\wedge d\bar z^\nu$ $\quad \psi_{\bar\mu\bar\nu} = -\psi_{\bar\nu\bar\mu}$

$$\int \langle \psi, \bar\partial\varphi\rangle\, d\mathrm{vol} = \int \frac{1}{2} h^{\bar\mu\alpha} h^{\bar\nu\beta} e^\rho \psi_{\bar\mu\bar\nu}\overline{(\bar\partial_\alpha\varphi_{\bar\beta} - \bar\partial_\beta\varphi_{\bar\alpha})}\, d\mathrm{vol}$$

$$= \int h^{\bar\mu\alpha}h^{\bar\nu\beta} e^\rho \psi_{\bar\mu\bar\nu}\,\partial_\alpha(\overline{\varphi_{\bar\beta}})\, \det(h)\, dx^1\cdots dy^n$$

$$= \int -\partial_\alpha(h^{\bar\mu\alpha}h^{\bar\nu\beta}e^\rho\psi_{\bar\mu\bar\nu}\det(h))\,\overline{\varphi_{\bar\beta}}\, dx^1\cdots dy^n$$

$$\partial_\alpha h^{\bar\mu\alpha} = -h^{\bar\mu\gamma}\Gamma^\alpha_{\alpha\gamma} = -h^{\bar\mu\alpha}\Gamma^\gamma_{\alpha\gamma} = -h^{\bar\mu\alpha}\det{}^{-1}(h)\,\partial_\alpha\det(h)$$

$$\partial_\alpha e^\rho = e^\rho\partial_\alpha\rho$$

$$\partial_\alpha h^{\bar\nu\beta} = h^{\bar\nu\gamma}\Gamma^\beta_{\alpha\gamma}$$

$h^{\bar\mu\alpha}h^{\bar\nu\gamma}\Gamma^\beta_{\alpha\gamma} \longrightarrow$ sym in $\bar\mu, \bar\nu$

$\psi_{\bar\mu\bar\nu}$: skew-sym in $\bar\mu,\bar\nu$

$$\Rightarrow (\bar\partial^*\psi)_{\bar\nu} = -h^{\bar\mu\alpha}(\partial_\alpha \psi_{\bar\mu\bar\nu} + (\partial_\alpha\rho)\psi_{\bar\mu\bar\nu})$$

$$(\bar\partial\varphi)_{\bar\mu\bar\nu} = \bar\partial_\mu \varphi_{\bar\nu} - \bar\partial_\nu \varphi_{\bar\mu}$$

$$\Rightarrow (\bar\partial^*\bar\partial\varphi)_{\bar\nu} = -h^{\bar\mu\alpha}(\partial_\alpha\bar\partial_\mu\varphi_{\bar\nu}^{①} - \partial_\alpha\bar\partial_\nu\varphi_{\bar\mu}^{②} + (\partial_\alpha\rho)(\bar\partial_\mu\varphi_{\bar\nu}^{③} - \bar\partial_\nu\varphi_{\bar\mu}^{④}))$$

ii) $f \in \Omega^0(L)$

$$\bar\partial f = \bar\partial_\alpha f \, d\bar z^\alpha$$

$$\int \langle \varphi, \bar\partial f\rangle \, dvol = \int e^\rho h^{\bar\mu\alpha} \varphi_{\bar\mu} \overline{(\bar\partial_\alpha f)} \det(h) \, dx^1 \cdots dy^n$$

$$= -\int \partial_\alpha (e^\rho \underline{h^{\bar\mu\alpha}} \varphi_{\bar\mu} \underline{\det(h)}) \bar f \, dx^1 \cdots dy^n$$

contribution cancels, as before

$$\Rightarrow \bar\partial^*\varphi = -h^{\bar\mu\alpha}(\partial_\alpha\varphi_{\bar\mu} + (\partial_\alpha\rho)\varphi_{\bar\mu})$$

$$\Rightarrow (\bar\partial\bar\partial^*\varphi)_{\bar\nu} = -h^{\bar\mu\alpha}(\partial_{\bar\nu}\partial_\alpha\varphi_{\bar\mu} + (\partial_{\bar\nu}\partial_\alpha\rho)\varphi_{\bar\mu} + (\partial_\alpha\rho)(\bar\partial_\nu\varphi_{\bar\mu})) - (\bar\partial_\nu h^{\bar\mu\alpha})(\partial_\alpha\varphi_{\bar\mu} + (\partial_\alpha\rho)\varphi_{\bar\mu})$$

$$\bar\partial_\nu h^{\bar\mu\alpha} = \overline{(\partial_\nu h^{\bar\alpha\mu})} = -\overline{h^{\bar\alpha\gamma}\Gamma^\mu_{\nu\gamma}} = -h^{\bar\gamma\alpha}\bar\Gamma^\mu_{\nu\gamma}$$

$$+ h^{\bar\mu\alpha}\bar\Gamma^\gamma_{\nu\mu}(\partial_\alpha\varphi_{\bar\gamma} + (\partial_\alpha\rho)\varphi_{\bar\gamma})$$

$$\Rightarrow (\bar\partial\bar\partial^*\varphi)_{\bar\nu} = -h^{\bar\mu\alpha}\Big(\bar\partial_\nu\partial_\alpha\varphi_{\bar\mu}^{②} + (\partial_{\bar\nu}\partial_\alpha\rho)\varphi_{\bar\mu} + (\partial_\alpha\rho)(\bar\partial_\nu\varphi_{\bar\mu})^{④} - \bar\Gamma^\gamma_{\nu\mu}(\partial_\alpha\varphi_{\bar\gamma}^{⑤} + (\partial_\alpha\rho)\varphi_{\bar\gamma}^{⑥})\Big)$$

iii) $$\nabla^{0,1}\varphi = (\bar\partial_\mu\varphi_{\bar\nu} - \bar\Gamma^\varepsilon_{\mu\nu}\varphi_{\bar\varepsilon})\, d\bar z^\mu \otimes d\bar z^\nu$$

$$\zeta = \zeta_{\bar\alpha\bar\beta}\, d\bar z^\alpha \otimes d\bar z^\beta$$

$$\int \langle \zeta, \nabla^{0,1}\varphi\rangle\, dvol = \int h^{\bar\alpha\mu} h^{\bar\beta\nu} e^\rho \zeta_{\bar\alpha\bar\beta} \overline{(\bar\partial_\mu\varphi_{\bar\nu} - \bar\Gamma^\varepsilon_{\mu\nu}\varphi_{\bar\varepsilon})} \det(h)\, dx^1 \cdots dy^n$$

sym

$$((\nabla^{0,1})^*\zeta)_{\bar\beta} = -h^{\bar\alpha\mu}(\partial_\mu\zeta_{\bar\alpha\bar\beta} + (\partial_\mu\rho)\zeta_{\bar\alpha\bar\beta})$$

$$((\nabla^{0,1})^*\nabla^{0,1}\varphi)_{\bar\nu} = -h^{\bar\mu\alpha}(\partial_\alpha(\nabla^{0,1}\varphi)_{\bar\mu\bar\nu} + (\partial_\alpha\rho)(\nabla^{0,1}\varphi)_{\bar\mu\bar\nu})$$

$$= -h^{\bar\mu\alpha}\Big(\partial_\alpha\bar\partial_\mu\varphi_{\bar\nu}^{①} + (\partial_\alpha\bar\Gamma^\varepsilon_{\mu\nu})\varphi_{\bar\varepsilon} - \bar\Gamma^\varepsilon_{\mu\nu}\partial_\alpha\varphi_{\bar\varepsilon}^{⑤} + (\partial_\alpha\rho)(\bar\partial_\mu\varphi_{\bar\nu})^{③} - (\partial_\alpha\rho)\bar\Gamma^\varepsilon_{\mu\nu}\varphi_{\bar\varepsilon}^{⑥}\Big)$$

IV) <u>theorem</u> Kodaira Bochner formula for $\varphi \in \Omega^{0,1}(L)$

$$\left(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial - (\nabla^{0,1})^*\nabla^{0,1}\varphi\right)_{\bar\nu}$$

$$= -h^{\bar\mu\alpha}\underbrace{(\partial_{\bar\nu}\partial_\alpha\rho)}_{\partial_\alpha\partial_{\bar\nu}\rho \,:\, \text{curvature of } L}\varphi_{\bar\mu} + h^{\bar\mu\alpha}(\partial_\alpha\bar\Gamma^{\varepsilon}_{\mu\nu})\varphi_{\bar\varepsilon}$$

conjugate of $h^{\bar\alpha\mu}R^{\varepsilon}_{\nu\bar\alpha\mu} = h^{\bar\gamma\varepsilon}h^{\bar\alpha\mu}R_{\bar\gamma\nu\bar\alpha\mu}$

$$= h^{\bar\gamma\varepsilon}R_{\bar\gamma\nu}$$

$\overline{h^{\bar\gamma\varepsilon}R_{\bar\gamma\nu}} = h^{\bar\varepsilon\gamma}R_{\bar\nu\gamma} \longleftarrow$ Ricci curvature of (M,ω)

$$h^{\bar\mu\alpha}R_{\bar\nu\alpha}\varphi_{\bar\mu}$$

§III. Kodaira vanishing theorem

1° Pairing with φ, and integrate it over M

$$\Rightarrow \int_M |\bar\partial^*\varphi|^2 + |\bar\partial\varphi|^2 = \int_M |\nabla^{0,1}\varphi|^2 - h^{\bar\mu\alpha}h^{\bar\nu\beta}\begin{pmatrix}\partial_{\bar\nu}\partial_\alpha\rho \\ -R_{\bar\nu\alpha}\end{pmatrix}\varphi_{\bar\mu}\overline{\varphi_{\bar\beta}}\, e^{\rho}$$

Hence, if $-\partial_{\bar\nu}\partial_\alpha\rho + R_{\bar\nu\alpha}$ is positive definite

(i.e. $(\partial_{\bar\nu}\partial_\alpha\rho - R_{\bar\nu\alpha})\, dz^\alpha\otimes d\bar z^\nu$ is a Hermitian metric on M)

then $H^{0,1}(L) = 0$

pf: $H^{0,1}(L) \cong \ker(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial) = \ker\bar\partial^* \cap \ker\bar\partial$

$\Rightarrow \int |\nabla^{0,1}\varphi|^2 + \text{positive definite}(\varphi,\varphi) = 0$

$\Rightarrow \varphi \equiv 0 \equiv \nabla^{0,1}\varphi$ ※

2° The similar argument works for $H^{0,q>0}(L)$ as well.

How to use it? Suppose that $L \to M$ is a "positive" line bundle. That is to say, it admits a Hermitian metric such that the first Chern form, $\frac{i}{2\pi}F^\nabla$ is everywhere positive definite

Then, for $L^{\otimes m}$, the curvature becomes m-mutiple

$\Rightarrow$ Kodaira vanishing : $H^{0,q}(L^{\otimes m}) = 0$ for $q \geq 0$, $m >> 1$

$\Rightarrow$ Hirzebruch-Riemann-Roch :

$H^0(L^{\otimes m}) = \{$global holomorphic section of $L^{\otimes m}\}$

has $\dim = \int_M ch(L^{\otimes m}) \, Td(T)$

$$ch(L^{\otimes m}) = \sum_{j=0} \frac{(m\, c_1(L))^j}{j!}, \qquad Td(T) = 1 + \cdots$$

$$N(m,n) := \frac{m^n}{n!} \int_M c_1(L)^n + O(m^{n-1}) \quad \text{for } m >> 1$$

(... more work ...) Kodaira embedding

For $m >> 1$, basis for $H^0(L^{\otimes m})$ gives a holomorphic embedding of M into $\mathbb{P}(H^0(L^{\otimes m})) = \mathbb{CP}^{N(m,n)-1}$

(admits a "positive" line bundle $\Longleftrightarrow$ projective)