§I. some linear algebra

$$V_{-1} \underset{S^*}{\overset{S}{\rightleftarrows}} V \underset{T^*}{\overset{T}{\rightleftarrows}} V_1 \qquad T \circ S = 0 \qquad \text{consider } \ker T / \operatorname{im} S$$

Suppose they all come with a Hermitian inner product.

$$V / \operatorname{im} S \cong (\operatorname{im} S)^\perp \Rightarrow \ker T / \operatorname{im} S \cong \ker T \cap (\operatorname{im} S)^\perp$$

Note that $(\operatorname{im} S)^\perp = \{ v : \langle v, Sw \rangle = 0 \quad \forall w \in V_{-1} \}$

$$\Leftrightarrow \langle S^* v, w \rangle = 0 \quad \forall w \in V_{-1}$$

$$\Leftrightarrow v \in \ker S^*$$

$$= \ker S^*$$

$$= \ker SS^* \qquad (SS^* v = 0,$$

Similarly, $\ker T = \ker T^* T$ $\qquad \langle SS^* v, v \rangle = 0 = |S^* v|^2)$

Hence. $\ker T / \operatorname{im} S \cong \ker SS^* \cap \ker T^* T$

<u>advantage</u> SS^* and $T^* T : V \circlearrowleft$

Moreover, if $(SS^* + T^* T) v = 0 \Rightarrow |S^* v|^2 + |Tv| = 0$

$$\Rightarrow S^* v = 0 = Tv$$

$$\Rightarrow \ker T / \operatorname{im} S \cong \ker (SS^* + T^* T)$$

We will use the infinite dimensional version of this

§II. Hermitian inner product on forms

$$\omega = \frac{i}{2} h_{\alpha \bar\beta} \, dz^\alpha \wedge d\bar z^\beta$$

On $\mathbb{C}^n$, $\omega_{std} = \frac{i}{2} dz^\alpha \wedge d\bar z^\alpha = dx^\alpha \wedge dy^\alpha$, $(\omega_{std})^n / n! = dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n$

In general, $\omega^n / n! = \det(h_{\alpha\bar\beta}) \, dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n =: d\mathrm{vol}$

1° $\Omega^{p,q} = \{ \text{sections of } \Lambda^p T^* \otimes \Lambda^q \bar{T}^* \}$

For
$$\varphi = \frac{1}{p! q!} \varphi_{\alpha_1 \cdots \alpha_p \bar\beta_1 \cdots \bar\beta_q} dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar z^{\beta_1} \wedge \cdots \wedge d\bar z^{\beta_q}$$

$$\psi = \frac{1}{p! q!} \psi_{\alpha_1 \cdots \alpha_p \bar\beta_1 \cdots \bar\beta_q} dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar z^{\beta_1} \wedge \cdots \wedge d\bar z^{\beta_q}$$

Define $\langle \varphi, \psi \rangle|_z = \frac{1}{p!q!} h^{\bar\mu_1\alpha_1} \cdots h^{\bar\mu_p\alpha_p} h^{\bar\beta_1\nu_1} \cdots h^{\bar\beta_q\nu_q} \varphi_{\alpha\bar\beta} \overline{\psi_{\mu\bar\nu}}$

(example $\Omega^{2,0}$ $\quad \varphi = \frac{1}{2}(\varphi\, dz^1\wedge dz^2 - \varphi\, dz^2\wedge dz^1) = \varphi\, dz^1\wedge dz^2$
$\psi = \frac{1}{2}(\psi\, dz^1\wedge dz^2 - \psi\, dz^2\wedge dz^1) = \psi\, dz^1\wedge dz^2$
$\langle\varphi,\psi\rangle = \frac{1}{2}(\varphi\bar\psi\, h^{\bar 1 1}h^{\bar 2 2} - \varphi\bar\psi\, h^{\bar 2 2}h^{\bar 1 1}) = h^{\bar 1 1}h^{\bar 2 2}\varphi\bar\psi$)

Then, $\langle\varphi,\psi\rangle = \int_M \langle\varphi,\psi\rangle_z\, d\mathrm{vol}_z$ is a Hermitian inner product on $\Omega^{p,q}$ ← ∞-dimensional

(Assume M is compact)

2° (L^2-) adjoint of $\bar\partial : \Omega^{p,q} \longrightarrow \Omega^{p,q+1}$

It is easier to describe $\bar\partial^*$ in terms of the metric dual

$T \longleftrightarrow \bar T^*$, $\quad \bar T \longleftrightarrow T^*$ $\quad \rightsquigarrow \bar\partial^* : \Lambda^p \bar T \otimes \Lambda^{q+1} T \longrightarrow \Lambda^p \bar T \otimes \Lambda^q T$

$h^{\beta\bar\alpha}\frac{\partial}{\partial z^\beta} \longleftrightarrow d\bar z^\alpha$, $\quad h^{\alpha\bar\beta}\frac{\partial}{\partial \bar z^\beta} \longleftrightarrow dz^\alpha$

Omit the "p"-parts for simplicity

$\varphi = \frac{1}{q!}\varphi_{\bar\beta_1\cdots\bar\beta_q}\, d\bar z^{\beta_1}\wedge\cdots\wedge d\bar z^{\beta_q}$.

$\bar\partial\varphi = \frac{1}{q!}(-1)^p\, \bar\partial_{\beta_0}\varphi_{\bar\beta_1\cdots\bar\beta_q}\, d\bar z^{\beta_0}\wedge d\bar z^{\beta_1}\wedge\cdots\wedge d\bar z^{\beta_q}$ turn coefficients alternating

$= \frac{(-1)^p}{q!}\frac{1}{q+1}\underbrace{(\bar\partial_{\beta_0}\varphi_{\bar\beta_1\cdots\bar\beta_q} - \bar\partial_{\beta_1}\varphi_{\bar\beta_0\hat{\bar\beta}_1\cdots\bar\beta_q} \pm \cdots)}\, d\bar z^{\beta_0}\wedge\cdots\wedge d\bar z^{\beta_q}$

$(\bar\partial\varphi)_{\bar\beta_0\cdots\bar\beta_q} = \sum_{j=0}^{q}(-1)^j\,\bar\partial_{\beta_j}\varphi_{\bar\beta_0\cdots\hat{\bar\beta}_j\cdots\bar\beta_q}$

$\psi = \frac{1}{(q+1)!}\psi_{\bar\mu_0\bar\mu_1\cdots\bar\mu_q}\, d\bar z^{\mu_0}\wedge\cdots\wedge d\bar z^{\mu_q}$

$\xrightarrow{\text{dual}} \frac{1}{(q+1)!}\underbrace{h^{\bar\mu_0\beta_0}\cdots h^{\bar\mu_q\beta_q}\psi_{\bar\mu_0\cdots\bar\mu_q}}_{\psi^{\beta_0\beta_1\cdots\beta_q}}\frac{\partial}{\partial z^{\beta_0}}\wedge\cdots\wedge\frac{\partial}{\partial z^{\beta_q}}$

$\Rightarrow \langle\bar\partial\varphi,\psi\rangle = \frac{(-1)^p}{(q+1)!}\int(\bar\partial\varphi)_{\bar\beta_0\cdots\bar\beta_q}\overline{\psi^{\beta_0\cdots\beta_q}}\,\det(h)\,dx^1\wedge\cdots\wedge dy^n$

$= \frac{(-1)^p}{(q+1)!}\int\sum_{j=0}\bar\partial_{\beta_j}\varphi_{\bar\beta_0\cdots\hat{\bar\beta}_j\bar\beta_q}\overline{\psi^{\beta_0\cdots\beta_q}}\,\det(h)\,dx^1\wedge\cdots\wedge dy^n$

Note that $-\bar\partial_{\beta_1}\varphi_{\bar\beta_0\bar\beta_2\cdots\bar\beta_q}\overline{\psi^{\beta_0\beta_1\cdots\beta_q}}$
$= -\bar\partial_{\beta_0}\varphi_{\bar\beta_1\bar\beta_2\cdots\bar\beta_q}\overline{\psi^{\beta_1\beta_0\cdots\beta_q}} = (j=0)$ $\quad$ (reason: ψ is alternating)

$$\Rightarrow \langle \bar\partial\varphi, \psi\rangle = \frac{(-1)^p}{q!}\int \bar\partial_{\beta_0}\varphi_{\bar\beta_1\cdots\bar\beta_q}\overline{\psi^{\beta_0\cdots\beta_q}}\det(h)\,dx^1\cdots dy^n$$

$$= \frac{(-1)^{p+1}}{q!}\int \varphi_{\bar\beta_1\cdots\bar\beta_q}\overline{\partial_{\beta_0}(\psi^{\beta_0\cdots\beta_q}\det(h))}\,dx^1\cdots dy^n$$

$$(\bar\partial^*\psi)^{\beta_1\cdots\beta_q} = (-1)^{p+1}\partial_{\beta_0}(\psi^{\beta_0\cdots\beta_q}\det(h))\frac{1}{\det(h)}$$

$$= (-1)^{p+1}\left(\partial_{\beta_0} + \partial_{\beta_0}(\log\det(h))\right)(\psi^{\beta_0\beta_1\cdots\beta_q})$$

<u>prop</u> For $\psi \in \Omega^{p,q}$

$$(\bar\partial^*\psi)^{\bar\alpha_1\cdots\bar\alpha_p\beta_1\cdots\beta_q} = (-1)^{p+1}\sum_{\beta_0}\left(\partial_{\beta_0} + \partial_{\beta_0}(\log\det h)\right)\psi^{\bar\alpha_1\cdots\bar\alpha_p\beta_0\beta_1\cdots\beta_q}$$

<u>remark</u> This formula does not use the Kähler condition.

§ III. on Kähler manifold

<u>goal</u> relate $\bar\partial^*$ to ∂ (and ∂^* to $\bar\partial$) through ∇

1° $\Omega^{p,q} \ni \varphi = \varphi_{\alpha_1\cdots\alpha_p\bar\beta_1\cdots\bar\beta_q}\,dz^{\alpha_1}\wedge\cdots\wedge dz^{\alpha_p}\wedge d\bar z^{\beta_1}\wedge\cdots\wedge d\bar z^{\beta_q} \,/\, p!\,q!$

The $dz^{\alpha_1}\wedge\cdots\wedge dz^{\alpha_p}\wedge d\bar z^{\beta_1}\wedge\cdots\wedge d\bar z^{\beta_q}$ component of $\nabla_\alpha\varphi$ is

$$(\nabla_\alpha\varphi)_{\alpha_1\cdots\alpha_p\bar\beta_1\cdots\bar\beta_q} = \partial_\alpha\varphi_{\alpha_1\cdots\alpha_p\bar\beta_1\cdots\bar\beta_q} - \sum_{j=1}^{p}\Gamma^{\gamma}_{\alpha\alpha_j}\varphi_{\alpha_1\cdots\hat\alpha_j\gamma\cdots\alpha_p\bar\beta_1\cdots\bar\beta_q}$$

Since $-\Gamma^{\gamma}_{\alpha\alpha_j}\varphi_{\alpha_1\cdots\hat\alpha_j\gamma\cdots\alpha_p\bar\beta_1\cdots\bar\beta_q}\,dz^{\alpha}\wedge dz^{\alpha_1}\wedge\cdots\wedge dz^{\alpha_j}\wedge\cdots\wedge dz^{\alpha_p}\wedge d\bar z^{\beta_1}\wedge\cdots\wedge d\bar z^{\beta_q} = 0$

sym　　　　skew-sym

$$\partial\varphi = (\nabla_\alpha\varphi)_{\alpha_1\cdots\alpha_p\bar\beta_1\cdots\bar\beta_q}\,dz^\alpha\wedge dz^{\alpha_1}\wedge\cdots\wedge dz^{\alpha_p}\wedge d\bar z^{\beta_1}\wedge\cdots\wedge d\bar z^{\beta_q}$$

(relate $\partial\varphi$ to $\nabla\varphi$)

Similarly, $\bar\partial\psi = (\bar\nabla_\beta\psi)_{\alpha_1\cdots\alpha_p\bar\beta_1\cdots\bar\beta_q}\,d\bar z^\beta\wedge dz^{\alpha_1}\wedge\cdots\wedge dz^{\alpha_p}\wedge d\bar z^{\beta_1}\wedge\cdots\wedge d\bar z^{\beta_q}$

<u>remark</u> In general, $d = \mathrm{alt}(\nabla^{LC} : \Omega^p \to \Omega^1\otimes\Omega^q)$

2° $h^{\bar\mu\nu}\psi_{\bar\mu} = \psi^\nu$　　Since the Hermitian inner product is parallel

$\bar T^*$　　T　　$h^{\bar\mu\nu}(\nabla_\gamma\psi)_{\bar\mu} = (\nabla_\gamma\psi)^\nu$

$(h^{\mu\bar\nu}$ " $\partial_\gamma\psi_{\bar\mu}$　　" $\partial_\gamma\psi^\nu + \Gamma^\nu_{\gamma\alpha}\psi^\alpha)$

In particular, $(\nabla_\beta\psi)^\beta = h^{\bar\mu\beta}(\nabla_\beta\psi)_{\bar\mu}$

$$(\bar\partial^*\psi)^{\bar\alpha_1\cdots\bar\alpha_p\beta_1\cdots\beta_q} = (-1)^{p+1}\sum_{\beta_0}\left(\partial_{\beta_0}+\partial_{\beta_0}(\log\det h)\right)\psi^{\bar\alpha_1\cdots\bar\alpha_p\beta_0\beta_1\cdots\beta_q}$$

Compare it with $(\nabla_{\beta_0}\psi)^{\bar A\,\beta_0\beta_1\cdots\beta_q}$

$$= \partial_{\beta_0}\psi^{\bar A\,\beta_0\beta_1\cdots\beta_q} + \Gamma^{\beta_0}_{\beta_0\gamma}\psi^{\bar A\,\gamma\beta_1\cdots\beta_q} + \sum_{j=1}^{q}\Gamma^{\beta_j}_{\beta_0\gamma}\psi^{\bar A\,\beta_0\beta_1\cdots\hat\beta_j\gamma\cdots\beta_q}$$

skew-sym $\quad 0$

sym

$$\Gamma^{\beta_0}_{\beta_0\gamma} = \Gamma^{\beta_0}_{\gamma\beta_0} = h^{\bar\varepsilon\beta_0}\partial_\gamma(h_{\beta_0\bar\varepsilon}) = \partial_\gamma(\log\det h)$$

$$\Rightarrow (\bar\partial^*\psi)^{\bar\alpha_1\cdots\bar\alpha_p\beta_1\cdots\beta_q} = (-1)^{p+1}(\nabla_{\beta_0}\psi)^{\bar\alpha_1\cdots\bar\alpha_p\beta_0\beta_1\cdots\beta_q}$$

$$\Rightarrow (\bar\partial^*\psi)_{\alpha_1\cdots\alpha_p\bar\beta_1\cdots\bar\beta_q} = (-1)^{p+1}h^{\bar\mu\beta_0}(\nabla_{\beta_0}\psi)_{\alpha_1\cdots\alpha_p\bar\mu\bar\beta_1\cdots\bar\beta_q}$$

3° So far, components of $\bar\partial^*\psi$ and $\partial\psi$ can be expressed in terms of $(\nabla_{\beta_0}\psi)$.

$\bar\partial^*: \Omega^{p,q+1}\longrightarrow\Omega^{p,q}$

$\partial: \Omega^{p,q+1}\longrightarrow\Omega^{p+1,q+1}$

To relate $\bar\partial^*$ with ∂, need a map from $\Omega^{p+1,q+1}$ to $\Omega^{p,q}$

Consider the "partial inverse" of $\wedge\omega: \Omega^{p,q}\longrightarrow\Omega^{p+1,q+1}$ defined as follows.

$$\varphi = \frac{1}{(p+1)!(q+1)!}\varphi_{\alpha_0\cdots\alpha_p\bar\beta_0\cdots\bar\beta_q}dz^{\alpha_0}\wedge\cdots\wedge dz^{\alpha_p}\wedge d\bar z^{\beta_0}\wedge\cdots\wedge d\bar z^{\beta_q}$$

lambda λ

$$\Lambda\varphi = \frac{1}{p!q!}(-1)^p\,i\,h^{\bar\beta_0\alpha_0}\varphi_{\alpha_0\cdots\alpha_p\bar\beta_0\cdots\bar\beta_q}dz^{\alpha_1}\wedge\cdots\wedge dz^{\alpha_p}\wedge d\bar z^{\beta_1}\wedge\cdots\wedge d\bar z^{\beta_q}$$

$\alpha_1\cdots\alpha_p\bar\beta_0 = (-1)^p\,\bar\beta_0\alpha_0\cdots\alpha_p$

<u>lemma</u> Λ is a real operator. $\overline{\Lambda\varphi} = \Lambda\bar\varphi$

pf:
$$\bar\varphi = \frac{(-1)^{(p+1)(q+1)}}{(p+1)!(q+1)!}\overline{\varphi_{\alpha_0\cdots\alpha_p\bar\beta_0\cdots\bar\beta_q}}dz^{\beta_0}\wedge\cdots\wedge dz^{\beta_q}\wedge d\bar z^{\alpha_0}\wedge\cdots\wedge d\bar z^{\alpha_p}$$

$$\Rightarrow \Lambda\bar\varphi = \frac{(-1)^{pq+p+1}}{p!q!}\,i\,h^{\bar\alpha_0\beta_0}\overline{\varphi_{\alpha_0\cdots\alpha_p\bar\beta_0\cdots\bar\beta_q}}dz^{\beta_1}\wedge\cdots\wedge dz^{\beta_q}\wedge d\bar z^{\alpha_1}\wedge\cdots\wedge d\bar z^{\alpha_p}$$

$$= \frac{(-1)^p}{p!q!}\overline{i\,h^{\bar\beta_0\alpha_0}\varphi_{\alpha_0\cdots\alpha_p\bar\beta_0\cdots\bar\beta_q}}\;\overline{dz^{\alpha_1}\wedge\cdots\wedge dz^{\alpha_p}\wedge d\bar z^{\beta_1}\wedge\cdots\wedge d\bar z^{\beta_q}}$$

✳

4° <u>prop</u> (Kähler identities)
$$[\partial,\Lambda] = i\bar\partial^*$$
$$[\bar\partial,\Lambda] = -i\partial^*$$
conjugate

pf. $\varphi \in \Omega^{p,q+1}$, $\qquad B = \beta_1 \cdots \beta_q$

$$(\partial\varphi)_{\alpha_0\cdots\alpha_p\bar\beta_0\cdots\bar\beta_q} = (\nabla_{\alpha_0}\varphi)_{\alpha_1\cdots\alpha_p\bar\beta_0\bar B} - (\nabla_{\alpha_1}\varphi)_{\alpha_0\alpha_2\cdots\alpha_p\bar\beta_0\bar B} \pm \cdots$$

$$(\Lambda\partial\varphi)_{\alpha_1\cdots\alpha_p\bar\beta_1\cdots\bar\beta_q} = (-1)^p i\, h^{\bar\beta_0\alpha_0}(\partial\varphi)_{\alpha_0\cdots\alpha_p\bar\beta_0\cdots\bar\beta_q}$$

$$= (-1)^p i\, h^{\bar\beta_0\alpha_0}\left((\nabla_{\alpha_0}\varphi)_{\alpha_1\cdots\alpha_p\bar\beta_0\bar B} - (\nabla_{\alpha_1}\varphi)_{\alpha_0\alpha_2\cdots\alpha_p\bar\beta_0\bar B} \pm \cdots\right)$$

$$\underset{p,q}{(\partial}\underset{p-1,q}{\Lambda\varphi})_{\alpha_1\cdots\alpha_p\bar\beta_1\cdots\bar\beta_q} = (\nabla_{\alpha_1}(\Lambda\varphi))_{\alpha_2\cdots\alpha_p\bar B} - (\nabla_{\alpha_2}(\Lambda\varphi))_{\alpha_1\alpha_3\cdots\alpha_p\bar B} \pm \cdots$$

$$= (-1)^{p-1} i\, h^{\bar\beta_0\alpha_0}\left((\nabla_{\alpha_1}\varphi)_{\alpha_0\alpha_2\cdots\alpha_p\bar\beta_0\bar B} - (\nabla_{\alpha_2}\varphi)_{\alpha_0\alpha_1\cdots\alpha_p\bar\beta_0\bar B} \pm \cdots\right)$$

raising / lowering indices commutes with ∇.

$$\Rightarrow [\partial, \Lambda]\varphi = (-1)^{p+1} i\, h^{\bar\mu\nu}(\nabla_\nu\varphi)_{\alpha_1\cdots\alpha_p\bar\mu\bar\beta_1\cdots\bar\beta_q} = i\bar\partial^*\varphi$$

\#

5° Consider $\Delta = dd^* + d^*d$ $\qquad \Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overline{\Box} = \partial\partial^* + \partial^*\partial$

(of $\bar\partial$) $\Box = -i\,\bar\partial(\partial\Lambda - \Lambda\partial) - i(\partial\Lambda - \Lambda\partial)\bar\partial$ $\Rightarrow$ $\partial\bar\partial = -\bar\partial\partial$

(of ∂) $\overline{\Box} = i\,\partial(\bar\partial\Lambda - \Lambda\bar\partial) + i(\bar\partial\Lambda - \Lambda\bar\partial)\partial$

$$\Delta = (\partial+\bar\partial)(\partial^*+\bar\partial^*) + (\partial^*+\bar\partial^*)(\partial+\bar\partial)$$

$$= \Box + \overline{\Box} + \partial\bar\partial^* + \bar\partial\partial^* + \partial^*\bar\partial + \bar\partial^*\partial$$

$$i(\partial\bar\partial^* + \bar\partial^*\partial) = \partial(\partial\Lambda - \Lambda\partial) + (\partial\Lambda - \Lambda\partial)\partial = 0$$

Similarly $\bar\partial\partial^* + \partial^*\bar\partial = 0$

<u>theorem</u> (Kähler identities) $\Delta = 2\Box = 2\overline{\Box}$

§ IV. Hodge theory

M: Kähler, compact without boundary

1° Consider $H^k_{dR}(M;\mathbb{C})$ $(\cong H^k_{dR}(M;\mathbb{R}) \otimes \mathbb{C})$

$$= \ker d / \operatorname{im} d \cong \ker(dd^* + d^*d)$$

Defined $\mathcal{H}^k = \{\varphi \in \Omega^k \mid \Delta\varphi = 0\}$ harmonic forms

$\Rightarrow H^k_{dR}(M) \cong \mathcal{H}^k$ (subject to ∞-dim$\underline{\text{l}}$ business)

2° Since $\Delta = 2\Box$: not only from Ω^k ↺

actually from $\Omega^{p,q}$ ↺

$\Rightarrow$ Let $\mathcal{H}^{p,q} = \{\varphi \in \Omega^{p,q} \mid \Box\varphi = 0\}$

Then $\mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q}$

3° $\mathcal{H}^{p,q} \ni \varphi \longrightarrow \overline{\varphi} \in \Omega^{q,p}$

$\Box\varphi = 0 \Rightarrow \overline{\Box}\,\overline{\varphi} = 0 \Rightarrow \Box\overline{\varphi} = 0$

Hence, $\mathcal{H}^{q,p} \cong \overline{\mathcal{H}^{p,q}}$

4° $\omega = \frac{i}{2} h_{\alpha\bar{\beta}}\, dz^\alpha \wedge d\bar{z}^\beta$

$d\omega = 0 \Rightarrow \partial\omega = 0 = \bar{\partial}\omega$,

$\Lambda\omega = \text{constant} \Rightarrow \partial^*\omega = 0 = \bar{\partial}^*\omega$

Hence, ω is a harmonic $(1,1)$-form

Similarly. $d\omega^2 = 0 \Rightarrow \partial\omega^2 = 0 = \bar{\partial}\omega^2$

$\Lambda\omega^2 = \text{constant} \cdot \omega \Rightarrow \partial^*\omega^2 = 0 = \bar{\partial}^*\omega$

$\cdots \Rightarrow \omega^k$ is a harmonic (k,k)-form

Note that $\omega^n = n!\, d\mathrm{vol} > 0$ everywhere

Hence, $\dim \mathcal{H}^{k,k} \geq 1$ for $k = 0, 1, \cdots, n$

5° <u>Poincaré</u> <u>duality</u> $H^k_{dR} \times H^{2n-k}_{dR} \longrightarrow \mathbb{C}$

$a \,,\, b \longmapsto \int_M a \wedge b$

is non-degenerate

Indeed. $\mathcal{H}^{p,q} \times \mathcal{H}^{n-p,n-q} \longrightarrow \mathbb{C}$ is non-degenerate

Hence. $(\mathcal{H}^{p,q})^* = \mathcal{H}^{n-p,n-q}$

(Usually done by introducing the Hodge $*$,

which we omit here)

6° Summary For a compact, n-dim'l Kähler manifold M

denote $\dim H^k_{dR}(M;\mathbb{C}) = \dim \mathcal{H}^k$ by h^k

and $\dim \mathcal{H}^{p,q}$ by $h^{p,q}$

Note that $0 \le p, q \le n$

(all dimensions are over $\mathbb{C}$)

They are usually drew as the "Hodge diamond"

h^0 $\quad h^{0,0}$

h^1 $\quad h^{1,0} \quad h^{0,1}$

h^2 $\quad h^{2,0} \quad h^{1,1} \quad h^{0,2}$

$\vdots$

h^{n-1} $\quad h^{n-1,0}$

h^n $\quad h^{n,0} \quad h^{n-1,1} \quad \cdots \quad h^{0,n}$

h^{n+1} $\quad h^{n,1}$

$\vdots$

h^{2n} $\quad h^{n,n}$

- $h^k = \sum_{p+q=k} h^{p,q}$
- $h^{2k} \ge h^{k,k} > 1$
- $h^{p,q} = h^{q,p} = h^{n-p,n-q} = h^{n-q,n-p}$
- $h^{2k+1} = \sum_{p+q=2k+1} h^{p,q}$ is even

topological constraint for Kähler manifolds.

§V. ellipticity (very sketchy)

1° The key ingredient for proving the Hodge theory is that Δ is elliptic

Δ acts on $\Omega^{p,q}$ as $2\left(\sum_j \frac{\partial}{\partial z^j}\frac{\partial}{\partial \bar{z}^j} + \frac{\partial}{\partial \bar{z}^j}\frac{\partial}{\partial z^j}\right) + (\text{lower order terms})$

$$\sum_j \left(\frac{\partial}{\partial x^j}\right)^2 + \left(\frac{\partial^2}{\partial y^j}\right)^2 \cdot \mathbb{1}_{\Lambda^{p,q}}$$

$\rightsquigarrow$ "Fourier transform" of the leading order (2^{nd} order) part is $(-2\pi i)^2 \sum_j (\xi^j)^2 + (\xi^{n+j})^2 \cdot \mathbb{1}_{\Lambda^{p,q}}$

2° In general, $\mathcal{D} : \Gamma(E) \circlearrowleft$ a differential operator of order m

Locally, it is a $\underbrace{k \times k}_{= \operatorname{rank} E}$ matrix of differential operators

$\rightsquigarrow$ "Fourier transform" on the leading order part is

principle symbol matrix $\leftarrow$ a $k \times k$ matrix whose elements are homogeneous in ξ^i's of order k

$\hookrightarrow$ Fourier variable of x^i's

$\mathcal{D}$ is said to be elliptic if the matrix is invertible when $\xi \neq 0$

Note that Δ is elliptic

3° For elliptic $\mathcal{D}$, one can construct the "inverse operator" by Fourier inversion (inverse of the principle symbol matrix) ...

The operator is good in the suitable sense of functional analysis (a.k.a. ∞-dimensional linear algebra)

... $\ker \mathcal{D}$ is finite dimensional