§I. curvature of Kähler metrics

0° $\omega = \frac{i}{2} \sum_{\mu,\nu} h_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^\beta = \frac{i}{2} \partial\bar\partial f$ ← local

$(h_{\alpha\bar\beta} = \partial_\alpha \bar\partial_\beta f)$

$(d\omega = 0 \iff \partial_\gamma h_{\alpha\bar\beta} = \partial_\alpha h_{\gamma\bar\beta})$

$$\nabla_\gamma \frac{\partial}{\partial z^\alpha} = \Gamma^\mu_{\gamma\alpha} \frac{\partial}{\partial z^\mu}, \quad \Gamma^\mu_{\gamma\alpha} = (\partial_\gamma h_{\alpha\bar\beta}) h^{\bar\beta\mu}$$

1° curvature of $T = \{\frac{\partial}{\partial z^\alpha}\}$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu - \nabla_{\cancel{[\mu,\nu]} \to 0}) \frac{\partial}{\partial z^\alpha}$$

$$= \nabla_\mu \left(\Gamma^\beta_{\nu\alpha} \frac{\partial}{\partial z^\beta}\right) - (\mu \leftrightarrow \nu) = \left(\partial_\mu(\Gamma^\beta_{\nu\alpha}) + \Gamma^\gamma_{\nu\alpha}\Gamma^\beta_{\mu\gamma} - (\mu\leftrightarrow\nu)\right)\frac{\partial}{\partial z^\beta}$$

$$= \partial_\mu\left((\partial_\nu h_{\alpha\bar\xi}) h^{\bar\xi\beta}\right) + (\partial_\nu h_{\alpha\bar\xi}) h^{\bar\xi\gamma} (\partial_\mu h_{\gamma\bar\eta}) h^{\bar\eta\beta} - (\mu\leftrightarrow\nu)$$

$$= (\partial_\mu \partial_\nu h_{\alpha\bar\xi}) h^{\bar\xi\beta} - \cancel{(\partial_\nu h_{\alpha\bar\xi})(\partial_\mu h^{\bar\xi\beta})} - (\mu\leftrightarrow\nu) = 0$$

$$+ \cancel{(\partial_\nu h_{\alpha\bar\xi})(\partial_\mu h^{\bar\xi\beta})}$$

$$\nabla_{\bar\mu}\nabla_{\bar\nu} - \nabla_{\bar\nu}\nabla_{\bar\mu} - \nabla_{[\bar\mu,\bar\nu]} \frac{\partial}{\partial z^\alpha} = 0$$

The interesting component:

$$(\nabla_{\bar\mu}\nabla_\nu - \nabla_\nu\nabla_{\bar\mu} - \nabla_{[\bar\mu,\nu]}) \frac{\partial}{\partial z^\alpha} = \nabla_{\bar\mu}\left(\Gamma^\beta_{\nu\alpha}\frac{\partial}{\partial z^\beta}\right) = (\partial_{\bar\mu}\Gamma^\beta_{\nu\alpha})\frac{\partial}{\partial z^\alpha}$$

Denote $\partial_{\bar\mu}\Gamma^\beta_{\nu\alpha}$ by $R^\beta_{\alpha\bar\mu\nu}$. $[\nabla_{\bar\mu}, \nabla_\nu]\frac{\partial}{\partial z^\alpha} = R^\beta_{\alpha\bar\mu\nu}\frac{\partial}{\partial z^\beta}$

2° Let $R_{\bar\beta\alpha\bar\mu\nu} = \langle [\nabla_{\bar\mu}, \nabla_\nu]\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar z^\beta}\rangle$ → the Hermitian inner product notation: use bar on the second argument

$$= h_{\gamma\bar\beta} R^\gamma_{\alpha\bar\mu\nu}$$

<u>prop</u> $R_{\bar\beta\alpha\bar\mu\nu} = \bar\partial_\beta \partial_\alpha \bar\partial_\mu \partial_\nu f - h^{\gamma\bar\varepsilon}(\bar\partial_\varepsilon \partial_\alpha \partial_\nu f)(\partial_\gamma \bar\partial_\beta \bar\partial_\mu f)$

pf: $R^\gamma_{\alpha\bar\mu\nu} = \bar\partial_\mu \Gamma^\gamma_{\nu\alpha} = \bar\partial_\mu\left((\partial_\gamma h_{\alpha\bar\varepsilon}) h^{\bar\varepsilon\gamma}\right)$

$$= (\bar\partial_\mu \partial_\gamma h_{\alpha\bar\varepsilon}) h^{\bar\varepsilon\gamma} + (\partial_\gamma h_{\alpha\bar\varepsilon})(\bar\partial_\mu h^{\bar\varepsilon\gamma})$$

$$\Rightarrow h_{\gamma\bar\beta} R^\gamma_{\alpha\bar\mu\nu} = \bar\partial_\mu \partial_\gamma h_{\alpha\bar\beta} + h_{\gamma\bar\beta}(\partial_\gamma h_{\alpha\bar\varepsilon})(\bar\partial_\mu h^{\bar\varepsilon\gamma})$$

$$= \bar\partial_\mu \partial_\gamma h_{\alpha\bar\beta} - h^{\bar\varepsilon\gamma}(\partial_\gamma h_{\alpha\bar\varepsilon})(\bar\partial_\mu h_{\gamma\bar\beta})$$

$$= \bar\partial_\mu \partial_\gamma \bar\partial_\beta \partial_\alpha f - h^{\bar\varepsilon\gamma}(\partial_\gamma \partial_\alpha \bar\partial_\varepsilon f)(\bar\partial_\mu \bar\partial_\beta \partial_\gamma f)$$ ※

<u>Cor</u> i) $R_{\bar\beta\alpha\bar\mu\nu} = R_{\bar\mu\alpha\bar\beta\nu} = R_{\bar\mu\nu\bar\beta\alpha} = R_{\bar\beta\nu\bar\mu\alpha}$

ii) $\overline{R_{\bar\beta\alpha\bar\mu\nu}} = R_{\alpha\bar\beta\nu\bar\mu}$

pf: the proposition + f is real-valued ※

4° Define $R_{\bar\mu\nu} = h^{\alpha\bar\beta} R_{\bar\beta\alpha\bar\mu\nu} = R^{\alpha}{}_{\alpha\bar\mu\nu}$ the Ricci curvature

i) $R^{\alpha}{}_{\alpha\bar\mu\nu} = \bar\partial_\mu \Gamma^{\alpha}_{\nu\alpha} = \bar\partial_\mu \big(\underbrace{\partial_\nu (h_{\alpha\bar\varepsilon}) h^{\bar\varepsilon\alpha}}_{= \operatorname{tr}((\partial_\nu h) h^{-1}) = (\det h)^{-1} \partial_\nu (\det h)} \big)$

$= \bar\partial_\mu \partial_\nu \log \det h$

ii) $R_{\bar\mu\nu} = h^{\alpha\bar\beta} R_{\bar\beta\alpha\bar\mu\nu} = \overline{h^{\beta\bar\alpha}}\ \overline{R_{\bar\alpha\beta\bar\nu\mu}} = \overline{R_{\bar\nu\mu}}$: Hermitian

iii) $K_M = \Lambda^n_{\mathbb{C}} T^*$ ($n = \dim_{\mathbb{C}} M$) : the canonical bundle of M

local section $= \{ dz^1 \wedge \cdots \wedge dz^n \}$

K_M is a holomorphic $\mathbb{C}^1$-bundle over M

If M carries a Hermitian metric, it induces a Hermitian bundle metric on K_M by $|dz^1 \wedge \cdots \wedge dz^n|^2 = (\det(h))^{-1}$

According to the discussion last week.

$F^\nabla = \bar\partial \partial \log (\det(h))^{-1} = -\bar\partial \partial \log \det(h)$

$\Rightarrow R_{\bar\mu\nu}\, dz^\nu \wedge d\bar z^\mu = F^\nabla$ of K_M

iv) (M, ω) is Ricci flat. $\Rightarrow$ K_M is a flat $\mathbb{C}^1$-bundle

(Suppose that M is simply connected. K_M admits a parallel section $\Rightarrow$ K_M is holomorphically trivial)

On the contrary, if K_M is holomorphically trivial.

$\Leftrightarrow$ admits a nowhere vanishing holomorphic section

<u>conjecture</u> (Calabi 1954) if so, (M, ω) admits a Ricci-flat Kähler metric

<u>thm</u> (Yau 1978) Yes, for compact M.

§II. relation to Riemann curvature tensor

1° complex structure : Define $J : T_{\mathbb{R}}M \longrightarrow T_{\mathbb{R}}M$ by

$$\frac{\partial}{\partial x^\mu} \longmapsto \frac{\partial}{\partial y^\mu}$$

$$\frac{\partial}{\partial y^\mu} \longmapsto -\frac{\partial}{\partial x^\mu}$$

$T \cong T_{\mathbb{R}}M$ over $\mathbb{R}$

$u \longmapsto u + \overline{u}$ $\Rightarrow$ J is induced by $u \longmapsto iu$.

and hence well-defined.

<u>remark</u> Given $J : T_{\mathbb{R}}M \longrightarrow T_{\mathbb{R}}M$ with $J^2 = -\mathbb{1}$,

M must be of even (real) dimension

Such a structure is called an almost complex structure.

For $X, Y \in T_{\mathbb{R}}M$, one can define

$$N(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$$

the so-called Nijenhuis tensor.

<u>fact</u> (Newlander–Nirenberg 1957)

J comes from a complex manifold structure if and only if $N \equiv 0$

2° i) For $X \in T_{\mathbb{R}}M$ $\quad X - iJ(X) \in T$

$$J(X - iJ(X)) = J(X) + iX = i(X - iJ(X))$$

Similarly $J(X + iJ(X)) = -i(X - iJ(X))$

Namely, T is the $(+i)$-eigenspace of J (on $T_{\mathbb{R}}M \otimes \mathbb{C}$)

$\overline{T}$ is the $(-i)$-eigenspace of J

ii) $(dz \otimes d\bar{z} = \underbrace{(dx \otimes dx + dy \otimes dy)}_{g} + i \underbrace{dx \wedge dy}_{\omega})$

Let us start with g on $T_{\mathbb{R}}M$.

Associate the Hermitian metric by

$$g(X - iJ(X), Y + iJ(Y)) = g(X,Y) + g(J(X), J(Y))$$

$$(Y + iJ(Y))^T (X - iJ(X)) \qquad + i\big(g(X, J(Y)) - g(J(X), Y)\big)$$

Also, it vanishes on $g(X - iJ(X), Y - iJ(Y))$

upshot $\boxed{\text{think}}$ • g has to obey $g(JX, JY) = g(X,Y)$

($Re(e^{i\theta}\frac{\partial}{\partial z^\mu})$ has the same length)

• $\omega(X,Y) = g(JX, Y) = g(J^2X, JY) = -g(JY, X)$
$= -\omega(Y,X)$

3° Now, suppose (M,ω) is Kähler $\Rightarrow \nabla^C = \nabla^{LC}$

$\nabla g = 0 = \nabla\omega$

$$Z(\omega(X,Y)) = \cancel{(\nabla_Z\omega)}^{0}(X,Y) + \omega(\nabla_Z X, Y) + \omega(X, \nabla_Z Y)$$
$$= \boxed{g(J(\nabla_Z X), Y)} + g(JX, \nabla_Z Y)$$
$$= Z(g(JX), Y) = \cancel{(\nabla_Z g)}^{0}(JX, Y) + g((\nabla_Z J)(X), Y) + \boxed{g(J(\nabla_Z X), Y)} + g(JX, \nabla_Z Y)$$

$\Rightarrow \nabla J \equiv 0$

$(\nabla(J(X)) = J(\nabla X))$

Riemann curvature

Therefore, $g(R(-,-)JX, JY)$
$= g(J(R(-,-)X, JY) = g(R(-,-)X, Y)$

We also know the symmetry of $g(R(\underbrace{-,-}) \underbrace{-,-})$

This gives another proof of the Corollary

4° Ricci curvature: Suppose that $h_{\alpha\bar\beta} = \delta_{\alpha\beta}$ at p

At p, $R_{\bar\mu\nu} = R_{\bar\alpha\alpha\bar\mu\nu}$ curvature (extend by $\mathbb{C}$-linearity)

real Riemann curvature

$= \frac{1}{16} R_{(\alpha+iJ\alpha)(\alpha-iJ\alpha)(\mu+iJ\mu)(\nu-iJ\nu)}$ α mean $\frac{\partial}{\partial x^\alpha}$ here

$= \frac{-i}{8} R_{\alpha J\alpha(\mu+iJ\mu)(\nu-iJ\nu)}$

$= \frac{-i}{4} R_{\alpha J\alpha\mu\nu} - \frac{1}{4} R_{\alpha J\alpha\mu J\nu}$

$J\alpha\, J\nu\, J\alpha\, \mu$
$-J\alpha\, \mu\, J\nu\, J\alpha$
(Bianchi)

$= +\frac{i}{4} R_{\alpha\mu\nu J\alpha} + \frac{i}{4} R_{\alpha\nu J\alpha\mu}$
$- \frac{1}{4} R_{\alpha\mu J\nu J\alpha} - \frac{1}{4} R_{\alpha J\nu J\alpha\mu}$

$+J\alpha\,\nu\, J\alpha\,\mu$
$-J\alpha\,\mu\,\nu\, J\alpha$

$= -\frac{i}{4} R_{\alpha\mu J\nu\alpha} - \frac{i}{4} R_{J\alpha\mu J\nu J\alpha} = +\frac{i}{2} Ric(\mu, J\nu)$
$- \frac{1}{4} R_{\alpha\mu\nu\alpha} - \frac{1}{4} R_{J\alpha\mu\nu J\alpha}$ $\quad + \frac{1}{2} Ric(\mu,\nu)$

real & imaginary part of $R_{\bar\mu\nu}$