§I. tangent and cotangent

M : complex manifold $\{z^\alpha\}$: local coordinate

1° T is the vector bundle with local sections $\{\frac{\partial}{\partial z^\alpha}\}$

$$\xi^\alpha \frac{\partial}{\partial z^\alpha} = \xi^\alpha \frac{\partial w^\beta}{\partial z^\alpha}\frac{\partial}{\partial w^\beta} = \eta^\beta \frac{\partial}{\partial w^\beta} \qquad \left(\frac{\partial \bar{w}^\beta}{\partial z^\alpha} = 0\right)$$

$$\Rightarrow g^T_{wz} = \left[\partial w^\beta / \partial z^\alpha\right]$$ $\beta\alpha$-entry β : row α : column

key feature. $g^T : \mathcal{U}_z \cap \mathcal{U}_w \longrightarrow Gl(n;\mathbb{C}) \subset \mathbb{C}^{n^2}$

is holomorphic

2° T^* is its $(\mathbb{C})$-dual bundle, which has $\{dz^\alpha\}$ as local sections. The transitions are holomorphic

3° $\overline{T}$, and $\overline{T}^*$ are the conjugate bundles

$\{\frac{\partial}{\partial \bar{z}^\alpha}\}$ $\{d\bar{z}^\alpha\}$ The transitions are anti-holomorphic

4° Consider complex valued differential forms on M

$$d\varphi = dx^\alpha \wedge \left(\frac{\partial}{\partial x^\alpha}\varphi\right) + dy^\alpha \wedge \left(\frac{\partial}{\partial y^\alpha}\varphi\right) \qquad z^\alpha = x^\alpha + i y^\alpha$$

in coefficient functions

$$\begin{cases} \frac{\partial}{\partial z^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} - i\frac{\partial}{\partial y^\alpha}\right) \\ \frac{\partial}{\partial \bar{z}^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} + i\frac{\partial}{\partial y^\alpha}\right) \end{cases}$$

$$dx^\alpha \wedge \frac{\partial}{\partial x^\alpha} + dy^\alpha \wedge \frac{\partial}{\partial y^\alpha}$$

$$= \frac{1}{2}(dz^\alpha + d\bar{z}^\alpha) \wedge \left(\frac{\partial}{\partial z^\alpha} + \frac{\partial}{\partial \bar{z}^\alpha}\right) + \frac{-i}{2}(dz^\alpha - d\bar{z}^\alpha) \wedge i\left(\frac{\partial}{\partial z^\alpha} - \frac{\partial}{\partial \bar{z}^\alpha}\right)$$

$$= \underbrace{dz^\alpha \wedge \frac{\partial}{\partial z^\alpha}}_{\partial :=} + \underbrace{d\bar{z}^\alpha \wedge \frac{\partial}{\partial \bar{z}^\alpha}}_{=: \bar{\partial}}$$

Note that these two operators are well-defined

(again, since $\frac{\partial \bar{w}^\beta}{\partial z^\alpha} = 0 = \frac{\partial w^\beta}{\partial \bar{z}^\alpha}$)

$\Rightarrow d = \partial + \bar{\partial}$

5° $T^*_{\mathbb{C}}M \ (= T^*M \otimes \mathbb{C}) = T^* \oplus \overline{T}^*$

dx^α, dy^α $\qquad dz^\alpha \qquad d\bar{z}^\alpha$

$$\Rightarrow \Lambda^k(T_{\mathbb{C}}^* M) = \bigoplus_{p+q=k} \left((\Lambda^p T^*) \otimes (\Lambda^q \overline{T}^*) \right)$$

denote its sections by $\Omega^{p,q}$ (e.g. $f\, dz^1 \wedge \dots \wedge dz^p \wedge d\bar{z}^1 \wedge \dots \wedge d\bar{z}^q$)

Note that $\partial : \Omega^{p,q} \longrightarrow \Omega^{p+1,q}$, $\bar\partial : \Omega^{p,q} \longrightarrow \Omega^{p,q+1}$

$$0 = d^2 = (\partial + \bar\partial)^2 = \partial^2 + (\partial\bar\partial + \bar\partial\partial) + \bar\partial^2$$

$$\Rightarrow \partial^2 = 0 = \bar\partial^2, \quad \partial\bar\partial = -\bar\partial\partial$$

§ II. Hermitian structure

1° Consider a Hermitian bundle metric on T

It is a bilinear form on $T \times \overline{T}$, hence a section of $T^* \otimes \overline{T}^*$

Write it as $h_{\alpha\bar\beta}\, dz^\alpha \otimes d\bar z^\beta$ $\begin{cases} h_{\alpha\bar\beta} = \overline{h_{\beta\bar\alpha}} : \text{Hermitian} \\ \text{only positive eigenvalues} \end{cases}$

2° real part : $\frac{1}{2}\left(h_{\alpha\bar\beta}\, dz^\alpha \otimes d\bar z^\beta + \overline{h_{\alpha\bar\beta}}\, d\bar z^\alpha \otimes dz^\beta \right)$

$$\overline{h_{\alpha\bar\beta}}\, d\bar z^\alpha \otimes dz^\beta = h_{\beta\bar\alpha}\, d\bar z^\alpha \otimes dz^\beta = h_{\alpha\bar\beta}\, d\bar z^\beta \otimes dz^\alpha$$

$$= h_{\alpha\bar\beta} \left(dz^\alpha \cdot d\bar z^\beta \right), \quad dz^\alpha \cdot d\bar z^\beta = \frac{1}{2}\left(dz^\alpha \otimes d\bar z^\beta + d\bar z^\beta \otimes dz^\alpha \right) \text{ symmetric product}$$

As explained last time, this defines a Riemannian metric on M ($T_{\mathbb{R}}M$)

3° imaginary part : $\frac{i}{2}\left(h_{\alpha\bar\beta}\, dz^\alpha \otimes d\bar z^\beta - h_{\alpha\bar\beta}\, d\bar z^\beta \otimes dz^\alpha \right)$

$$= \frac{i}{2} h_{\alpha\bar\beta} \left(dz^\alpha \wedge d\bar z^\beta \right), \quad dz^\alpha \wedge d\bar z^\beta = dz^\alpha \otimes d\bar z^\beta - d\bar z^\beta \otimes dz^\alpha$$

4° These are all equivalent data, we will mainly use

$$\omega = \frac{i}{2} h_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^\beta \in \Omega^{1,1}$$

(recall the discussion last time. complex structure $\Leftrightarrow$ J is given)

5° lemma Any complex manifold admits a Hermitian metric

pf: Partition of unity ...... ※

§ III. holomorphic vector bundle and Chern connection

Instead of studying connection on T. let us do the theory for holomorphic vector bundles

1° holomorphic vector bundle : admits trivialization cover such that transitions are holomorphic

local function on M, with value in $Gl(k;\mathbb{C}) \subset \mathbb{C}^k$

2° Suppose that $\{e_\mu\}$ and $\{\tilde{e}_\nu\}$ are two such trivializing sections. Namely. $\tilde{e}_\nu = g^\mu_\nu e_\mu$ g^μ_ν are holomorphic $(\bar\partial g^\mu_\nu \equiv 0)$

Hence. for a section $V^\mu e_\mu = W^\nu \tilde{e}_\nu$

we can define $\bar\partial(V^\mu e_\mu)$ by $(\bar\partial V^\mu) e_\mu$

In particular, $\bar\partial e_\mu \equiv 0$ if they induce the "holomorphic" trivialization

$$\bar\partial : \Gamma(E) \longrightarrow \Gamma(\overline{T}^* \otimes E) =: \Omega^{0,1}(E)$$

$$\Omega^{1,0}(E) \oplus \Omega^{0,1}(E)$$

$$\|$$

3° Recall: connection $\nabla : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E) \subset \Gamma(T^*_{\mathbb{C}}M \otimes E)$

extend by $\mathbb{C}$-linearity

prop Given a Hermitian metric on E, there exists a unique connection which is compatible with metric and whose $(0,1)$-component is $\bar\partial$

Chern connection

pf: $\nabla e_\mu = A^\nu_\mu \otimes e_\nu$ $A^\nu_\mu \in \Omega^{1,0}(\mathcal{U})$

$H_{\mu\bar\nu} = \langle e_\mu, e_\nu \rangle$

$dH_{\mu\bar\nu} = \langle \nabla e_\mu, e_\nu \rangle + \langle e_\mu, \nabla e_\nu \rangle$

$= \langle A^\varepsilon_\mu e_\varepsilon, e_\nu \rangle + \langle e_\mu, A^\varepsilon_\nu e_\varepsilon \rangle$

$$\partial H_{\mu\bar\nu} + \bar\partial H_{\mu\bar\nu} = A_\mu^\varepsilon H_{\varepsilon\bar\nu} + \bar A_\nu^\varepsilon H_{\mu\bar\varepsilon}$$

Due to the splitting $T^* \oplus \bar T^*$, $\partial H_{\mu\bar\nu} = A_\mu^\varepsilon H_{\varepsilon\bar\nu}$

$$\Rightarrow \begin{array}{c} \text{column} \to \\ \text{row} \to \end{array} A_\mu^\nu = (\partial H_{\mu\bar\varepsilon}) H^{\bar\varepsilon\nu}$$

$$(H^{\bar\mu\varepsilon} H_{\varepsilon\bar\nu} = \delta^{\bar\mu}_{\bar\nu} , \quad H_{\mu\bar\varepsilon} H^{\bar\varepsilon\nu} = \delta_\mu^\nu)$$

<u>Cor</u> The curvature of a Chern connection is of type (1,1)

pf: $\nabla(s^\mu e_\mu) = (ds^\nu) e_\nu + s^\mu A_\mu^\nu e_\nu$ ($\nu \leftarrow$ row, $\mu \leftarrow$ column) (a transpose issue here)

$$A = H^{-1}\partial H$$

$$F_A = d(H^{-1}\partial H) + H^{-1}(\partial H) \wedge H^{-1}(\partial H)$$

$$= (\partial H^{-1} + \bar\partial H^{-1}) \wedge \partial H + H^{-1}\bar\partial\partial H - (\partial H^{-1}) \wedge \partial H$$

$$= H^{-1}\bar\partial\partial H + \bar\partial H^{-1} \wedge \partial H$$

<u>remark</u> When E has rank 1 (complex line bundle)

H is a locally defined real value function

$$F = H^{-1}\bar\partial\partial H - H^{-2}\bar\partial H \wedge \partial H$$

$$\bar\partial\partial(\log H) = \bar\partial\left(\frac{\partial H}{H}\right) = F$$

§IV. the main example: Fubini-Study metric

1° Consider the tautological bundle over $\mathbb{CP}^n$

$$L \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$$

$$([Z_0, \cdots, Z_n], \vec{w}) \quad \vec{w} \,/\!/\, (Z_0, \cdots, Z_n]$$

Holomorphic trivialization $((z_1, \cdots, z_n), \xi) \longmapsto ([1, z_0, \cdots, z_n], (\xi, \xi z_1, \cdots, \xi z_n))$

Do this for other charts. transition $= z_j$: holomorphic

It means that $\xi \equiv 1$ is a local holomorphic section.

2° Hermitian metric: define by $|\vec{w}| = (1 + |z_1|^2 + \cdots + |z_n|^2)|\xi|^2$

In other words, $H = 1 + \sum_{j=1}^{n} |z_j|^2$

$$F = \bar\partial \partial \log H$$

$$\partial \log H = H^{-1} \sum_j \bar z_j \, dz_j$$

$$\bar\partial\partial \log H = -H^{-2}\Big(\sum_j H \, dz_j \wedge d\bar z_j - \sum_j \bar z_j dz_j \wedge \sum_k z_k d\bar z_k \Big)$$

$$= -H^{-2}\Big(\sum_j (H - |z_j|^2)\, dz_j \wedge d\bar z_j - \sum_{j\neq k} \bar z_j z_k \, dz_j \wedge d\bar z_k \Big)$$

$\boxed{\text{Check}}$ $-\frac{i}{2\pi} F$ corresponds to a Hermitian metric on $\mathbb{CP}^n$, which is called the Fubini-Study metric

§V. Kähler metric

$\omega = \overline{\omega}$

1° Focus on the real (1,1)-form associated to the Hermitian metric

$$\omega = \frac{i}{2} h_{\alpha\bar\beta} \, dz^\alpha \wedge d\bar z^\beta$$

$$= \frac{i}{2} h_{\alpha\bar\beta} (dx^\alpha + i\, dy^\alpha) \wedge (dx^\beta - i\, dy^\beta)$$

$$= \frac{i}{2} h_{\alpha\bar\beta} (dx^\alpha \wedge dx^\beta + dy^\alpha \wedge dy^\beta) + \frac{1}{2} h_{\alpha\bar\beta} (dx^\alpha \wedge dy^\beta - dy^\alpha \wedge dx^\beta)$$

$$h_{\alpha\bar\beta}\, dx^\alpha \wedge dx^\beta = \frac{1}{2}\big(h_{\alpha\bar\beta}\, dx^\alpha \wedge dx^\beta + \cancel{h_{\beta\bar\alpha}\, dx^\beta \wedge dx^\alpha} \big)$$

$$- \overline{h_{\alpha\bar\beta}}\, dx^\alpha \wedge dx^\beta$$

$$= -\frac{1}{2} (\mathrm{Im}\, h_{\alpha\bar\beta}) (dx^\alpha \wedge dx^\beta + dy^\alpha \wedge dy^\beta) + (\mathrm{Re}\, h_{\alpha\bar\beta})\, dx^\alpha \wedge dy^\beta$$

For the standard metric on $\mathbb{C}^n$, $\omega = \frac{i}{2} dz^\alpha \wedge d\bar z^\alpha \Rightarrow d\omega = 0$

For the Fubini-Study metric on $\mathbb{CP}^n$, $\omega = \frac{-i}{2\pi} F \Rightarrow d\omega = 0$

<u>defn</u> A Hermitian metric is called a Kähler metric if $d\omega = 0$

2° Consider the Chern connection on T

$$\nabla \frac{\partial}{\partial z^\alpha} = A_\alpha^\beta \frac{\partial}{\partial z^\beta} = (\partial h_{\alpha\bar\varepsilon})\, h^{\bar\varepsilon\beta} \otimes \frac{\partial}{\partial z^\beta}$$

Namely, $\nabla_\gamma \frac{\partial}{\partial z^\alpha} = (\partial_\gamma h_{\alpha\bar\varepsilon})\, h^{\bar\varepsilon\beta} \frac{\partial}{\partial z^\beta}$: $\Gamma^\beta_{\gamma\alpha}$

⚠ Just use same notation as the Levi-Civita

($\nabla_{\bar\gamma}$ in Kodaira's book)

Note that $\nabla_{\bar\gamma} \frac{\partial}{\partial z^\alpha} = 0$

Induced connection on T^*: $dz^\alpha\left(\frac{\partial}{\partial z^\beta}\right) = \delta^\alpha_\beta$

By requiring $(\nabla dz^\alpha)\left(\frac{\partial}{\partial z^\beta}\right) + (dz^\alpha)\left(\nabla \frac{\partial}{\partial z^\beta}\right) = d\delta^\alpha_\beta = 0$

$$\Rightarrow \begin{cases} \nabla_\gamma dz^\alpha = -\Gamma^\alpha_{\gamma\beta}\, dz^\beta \\ \nabla_{\bar\gamma} dz^\alpha = 0 \end{cases}$$

For $\bar T$, it is not a holomorphic bundle, but anti-holomorphic

Take the "conjugate" of the connection on T

$$\nabla \frac{\partial}{\partial z^\alpha} = (\partial h_{\alpha\bar\varepsilon})\, h^{\bar\varepsilon\beta} \otimes \frac{\partial}{\partial z^\beta} = \Gamma^\beta_{\gamma\alpha}\, dz^\gamma \otimes \frac{\partial}{\partial z^\beta}$$

$$\Rightarrow \nabla \frac{\partial}{\partial \bar z^\alpha} = \overline{\Gamma^\beta_{\gamma\alpha}}\, d\bar z^\gamma \otimes \frac{\partial}{\partial \bar z^\beta}$$ Note that $\nabla_\gamma \frac{\partial}{\partial \bar z^\alpha} = 0$

Similarly, for $\bar T^*$, $\begin{cases} \nabla_\gamma d\bar z^\alpha = 0 \\ \nabla_{\bar\gamma} d\bar z^\alpha = -\overline{\Gamma^\alpha_{\gamma\beta}}\, d\bar z^\beta \end{cases}$

3° lemma $h_{\alpha\bar\beta}\, dz^\alpha \otimes d\bar z^\beta$ is parallel

pf:
$$\nabla_\gamma (h_{\alpha\bar\beta}\, dz^\alpha \otimes d\bar z^\beta) = (\partial_\gamma h_{\alpha\bar\beta})\, dz^\alpha \otimes d\bar z^\beta - h_{\alpha\bar\beta}\, \Gamma^\alpha_{\gamma\varepsilon}\, dz^\varepsilon \otimes d\bar z^\beta = 0$$

$$(\partial_\gamma h_{\alpha\bar\beta})\, dz^\alpha = h_{\varepsilon\bar\beta} (\partial_\gamma h_{\alpha\bar\mu})\, h^{\bar\mu\varepsilon}\, dz^\alpha = h_{\varepsilon\bar\beta}\, \Gamma^\varepsilon_{\gamma\alpha}\, dz^\alpha$$

Similar computation for $\nabla_{\bar\gamma}$ ※

prop The metric is Kähler if and only if $\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}$

pf: $\omega = \frac{i}{2} h_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^\beta$

$$d\omega = \frac{i}{2}\left(\partial_\gamma h_{\alpha\bar\beta}\, dz^\gamma \wedge dz^\alpha \wedge d\bar z^\beta + \bar\partial_\gamma h_{\alpha\bar\beta}\, d\bar z^\gamma \wedge dz^\alpha \wedge d\bar z^\beta\right)$$
$$= \frac{i}{4}\Big((\partial_\gamma h_{\alpha\bar\beta} - \partial_\alpha h_{\gamma\bar\beta})\, dz^\gamma \wedge dz^\alpha \wedge d\bar z^\beta + (\bar\partial_\gamma h_{\alpha\bar\beta} - \bar\partial_\beta h_{\alpha\bar\gamma})\, d\bar z^\gamma \wedge dz^\alpha \wedge d\bar z^\beta \Big)$$

$$\partial_\gamma h_{\alpha\bar\beta} - \partial_\alpha h_{\gamma\bar\beta} = (\Gamma^\varepsilon_{\gamma\alpha} - \Gamma^\varepsilon_{\alpha\gamma}) h_{\varepsilon\bar\beta}$$

$$\bar\partial_\gamma h_{\alpha\bar\beta} - \bar\partial_\beta h_{\alpha\bar\gamma} = \overline{\partial_\gamma h_{\beta\bar\alpha} - \partial_\beta h_{\gamma\bar\alpha}} = \cdots$$

※

4° ∇ induces a connection on $T_{\mathbb{R}}M$
takes the form $u + \bar u$ for $u \in T$

$$\nabla_{u+\bar u}(V+\bar V) = \nabla_u V + \nabla_{\bar u}\bar V + \nabla_u \bar V + \nabla_{\bar u} V \ : \text{still "real"}$$

Since the Chern connection is compatible with the Hermitian metric, the induced connection on $T_{\mathbb{R}}M$ must be compatible with the induced Riemannian metric

Note that $\nabla_\gamma \frac{\partial}{\partial z^\alpha} - \nabla_\alpha \frac{\partial}{\partial z^\gamma} = (\Gamma^\beta_{\gamma\alpha} - \Gamma^\beta_{\alpha\gamma}) \frac{\partial}{\partial z^\beta}$

It follows that the metric is Kähler if and only if the Chern connection coincides with (the $\mathbb{C}$-linear extension of) the Levi-Civita connection

5° $X \subset (M, \omega)$ complex submanifold in a Kähler manifold
$\Rightarrow$ X is Kähler as well

pf: $\forall p \in X$, $\exists$ local coordinate for p in M, $\{z^\alpha\}_{\alpha=1}^n$
such that X is given by $\{z^{k+1} = \cdots = z^n = 0\}$

$$\omega = \frac{i}{2} \sum_{\alpha,\beta=1}^n h_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^\beta$$

$$\Rightarrow \omega|_X = \frac{i}{2} \sum_{\alpha,\beta=1}^k h_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^\beta$$

(linear algebra)

$$\begin{cases} [h_{\alpha\bar\beta}]_{1\le\alpha,\beta\le k} \text{ is still Hermitian and positive definite} \\ d(\omega|_X) = (d\omega)|_X = 0 \end{cases}$$

※

In particular, complex submanifolds in $\mathbb{CP}^n$ are Kähler.

§VI. Dolbeault lemma

recall Poincaré lemma : $H^k_{dR}(\mathcal{U}) = 0$ except $k=0$

Note that $\bar\partial^2 = 0$, we now examine the $\bar\partial$-version of it.

1° lemma $\mathcal{U} \subset \mathbb{C}^1$, $\alpha \in C^\infty(\mathcal{U}; \mathbb{C})$. Suppose that $\overline{B(p;r)} \subset \mathcal{U}$

Then, $f(z) = \frac{1}{2\pi i} \int_B \frac{\alpha(\xi, \bar\xi)}{\xi - z} d\xi \wedge d\bar\xi \in C^\infty(B; \mathbb{C})$

satisfies $\frac{\partial f}{\partial \bar z} = \alpha$, i.e. $\bar\partial f = \alpha d\bar z$

pf: Cauchy integral formula for smooth functions ...... ※

2° lemma $\mathcal{U} \subset \mathbb{C}^n$, $\alpha \in \Omega^{0,1}$ with $\bar\partial\alpha = 0$.

Suppose that $\overline{P(p;r)} \subset \mathcal{U}$. Then, there exists $f \in C^\infty(P; \mathbb{C})$

such that $\bar\partial f = \alpha$

pf: $\alpha = \sum_{\mu=1}^n \alpha_\mu d\bar z^\mu$ Do induction.

i) $\alpha = \alpha_1 d\bar z^1$, $\bar\partial\alpha = 0 \Rightarrow \alpha_1$ is holomorphic in $z_2, \cdots, z_n$

By the previous lemma. $\exists f \ni \partial f / \partial \bar z^1 = \alpha_1$

The construction implies that $\partial f / \partial \bar z^\mu = 0$ for $\mu > 1$.

ii) Assume true for $\mathrm{span}\{d\bar z^1, \cdots, d\bar z^k\}$

$\alpha = \alpha_{k+1} d\bar z^{k+1} + \tilde\alpha$

$\bar\partial\alpha = 0 \Rightarrow \alpha_{k+1}$ is holomorphic in $z_{k+2}, \cdots, z_n$

As above, $\alpha_{k+1} = \partial\tilde f / \partial \bar z^{k+1}$ where $\tilde f$ is holomorphic in $z_{k+2}, \cdots, z_n$

$\Rightarrow \alpha - \bar\partial\tilde f \in \mathrm{span}\{d\bar z^1, \cdots, d\bar z^k\}$

※

3° It is in generally true for $\Omega^{p,q-1} \xrightarrow{\bar\partial} \Omega^{p,q} \xrightarrow{\bar\partial} \Omega^{p,q+1}$

Now, back to Kähler forms

$d\omega = 0 \Rightarrow \omega$ is locally $d(\beta + \alpha)$ $\beta \in \Omega^{1,0}, \alpha \in \Omega^{0,1}$

Ω^1

$\beta \ni \Omega^{1,0} \oplus \Omega^{0,1} \ni \alpha$

$\Omega^{2,0} \oplus \underset{\omega}{\Omega^{1,1}} \oplus \Omega^{0,2}$

$\Rightarrow \partial\beta = 0, \bar\partial\alpha = 0, \bar\partial\beta + \partial\alpha = \omega$

Since $\omega = \bar\omega$, by replacing $\beta + \alpha$ by $\frac{1}{2}(\beta + \alpha) + (\bar\alpha + \bar\beta)$, we may assume $\beta = \bar\alpha$

Now, $\omega = \bar\partial\bar\alpha + \partial\alpha$, and $\alpha \in \Omega^{0,1}$ satisfying $\bar\partial\alpha = 0$

By the Dolbeault lemma. $\alpha = \bar\partial F = \bar\partial(\tilde f + i f)$

$$\partial\alpha = \partial\bar\partial(\tilde f + i f)$$

$$\bar\partial\bar\alpha = \bar\partial\partial(\tilde f - i f)$$

$\Rightarrow \omega = 2i\,\partial\bar\partial f$ locally

<u>example</u> i) We have seen this for the Fubini – Study metric

ii) On $\mathbb{C}^n$, $\omega = \frac{i}{2} dz^\mu \wedge d\bar z^\mu = \frac{i}{2}\partial\bar\partial \sum_\mu |z^\mu|^2$