§I. projective submanifold

1° $\{ Z_0^3 + Z_1^3 + Z_2^3 = 0 \} \subset \mathbb{CP}^2$

$d \hookrightarrow$ $3(Z_0^2\, dZ_0 + Z_1^2\, dZ_1 + Z_2^2\, dZ_2)$

If $Z_0 \neq 0$, plug in $Z_0 = 1$, $Z_1 = z_1$, $Z_2 = z_2$

$\rightsquigarrow$ $z_1^2\, dz_1 + z_2^2\, dz_2$: surjective on $\{1 + z_1^2 + z_2^2 = 0\}$

2° In general, given k homogeneous equations on $\mathbb{CP}^N = \mathbb{C}^{N+1} \setminus \{0\} / \mathbb{C}^*$

If their differential on $\mathbb{C}^{N+1}\setminus\{0\}$ is surjective $\mathbb{C}^k$ everywhere on the zero locus, then the zero locus is a $(N-k)$-dimensional complex submanifold in $\mathbb{CP}^N$

<u>key</u> homogeneous $\Rightarrow$ $\alpha Z^j \frac{\partial}{\partial Z^j}$ must in the kernel of the differential

$\Rightarrow$ surjectivity cannot come from the $\mathbb{C}^*$-action orbit

§II. Hopf manifold

$\mathbb{C}^N \setminus \{0\} / \mathbb{Z} = \langle g \rangle$ Choose $\alpha_j \in \mathbb{C}$ with $0 < |\alpha_j| < 1$ for $j = 1, \cdots, N$

$g^m \cdot (w_1, \cdots, w_N) = (\alpha_1^m w_1, \cdots, \alpha_N^m w_N)$

Note that $\sum_j |w_j|^2 = 1 \xrightarrow{g} \sum_j \alpha_j^{-1} |w_j|^2 = 1$

Consider $\Gamma_s = \left\{ \sum_{j=1}^n \alpha_j^s |w_j|^2 = 1 \right\}$ for $s \in \mathbb{R}_{\geq 0}$

$\Rightarrow g \cdot \Gamma_s = \Gamma_{s-1}$ each $\Gamma_s \cong S^{n-1}$

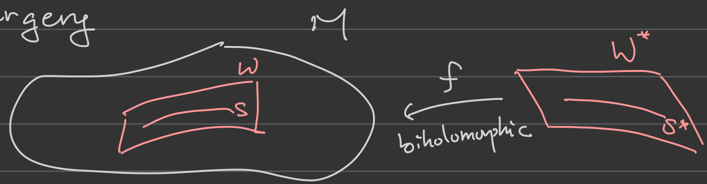
From this, the Hopf manifold $= \coprod_{s \in [-1,0]} \Gamma_s \Big/ \Gamma_0 \simeq \Gamma_{-1}$ by g

$\overset{\text{diffeomorphism}}{\cong} S^{n-1} \times S^1$ $\boxed{\text{think}}$

A priori, it is not clear that $S^{n-1} \times S^1$ is a complex manifold.

§III. Hirzebruch surface

0° surgery



$S \subset M$, submanifold, W: neighborhood of S

$\rightsquigarrow$ $M^* = (M \setminus S) \cup_f W^*$ identify $W \setminus S$ with $W^* \setminus S^*$

1° $M = \mathbb{CP}^1 \times \mathbb{CP}^1$

$S = \{[(0,1)]\} \times \mathbb{CP}^1$ $\qquad W^* = \{|z| < \varepsilon\} \times \mathbb{CP}^1$

$W = \{[(z,1)] : |z| < \varepsilon\} \times \mathbb{CP}^1$ $\qquad S^* = \{0\} \times \mathbb{CP}^1$

Fix $m \in \mathbb{Z}$

$f_m : W^* \setminus S^* \longrightarrow W \setminus S$

$(z, [(\zeta_0, \zeta_1)]) \mapsto ([z,1], [(\zeta_0, \zeta_1 z^m)])$: biholomorphic

Denote the outcome by M_m^*

<u>remark</u> Clearly, $M_0^* = M = \mathbb{CP}^1 \times \mathbb{CP}^1$

i) M_m^* is not biholomorphic to M_n^* for $m \neq \pm n$ — by studying holomorphic curves in them

ii) M_1^* is not homeomorphic to M_0^* (P.15)

iii) M_{even}^* is diffeomorphic to $M_0^* = M$

iv) M_{odd}^* is diffeomorphic to M_1^*

$\}$ based on I.4

$\Rightarrow$ There are different complex structures on $S^2 \times S^2$.

Note that the complex structure on S^2 is rigid.

2° M : complex manifold. E: <u>holomorphic</u> vector bundle over M

$\Leftrightarrow$ transition $U \cap V \to Gl(n;\mathbb{C}) \underset{\text{open}}{\subset} \mathbb{C}^{n^2}$ is holomorphic

Suppose that rank $E > 1$, we can consider its projectification

$$\mathbb{P}(E) = E \setminus \{M \text{ as zero section}\} \big/ \mathbb{C}^*$$

recall $L_n \longrightarrow \mathbb{CP}^1$ with transition $g_{vu}^n = w^n = z^{-n}$

Consider $\mathbb{P}(L_0 \oplus L_n)$

For $L_0 \oplus L_n$

$$(z, (\xi_0, \xi_1)) \sim (w, (\eta_0, \eta_1))$$
$$\qquad\qquad\qquad\qquad \| $$
$$\qquad\qquad\qquad (\xi_0, \xi_1 z^{-n}) \qquad [(\eta_0, \eta_1)]$$

$W^* \setminus S^*$ $\qquad\qquad$ $W \setminus S$ $\qquad\qquad \cap$

complement of $[(0,1)] \times \mathbb{CP}^1$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$

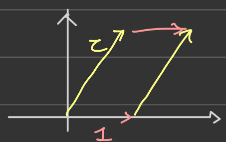
$\Rightarrow M_m^* = \mathbb{P}(L_0 \oplus L_{-m})$

We can also see it is $\mathbb{P}(L_m \oplus L_0) \Rightarrow M_m^* = M_{-m}^*$

§IV. log transform

0° $T = \mathbb{C}^1 / \mathbb{Z}^2$ is a torus (elliptic curve)

Fix $\tau \in \mathbb{C}$ with $\operatorname{im} \tau > 0$, then $\{1, \tau\}$: linearly independent over $\mathbb{R}$. consider $\zeta \sim \zeta + m + n\tau$



1° $M = \mathbb{CP}^1 \times T$

$S = \{[0,1]\} \times T$ $\qquad\qquad$ $S^* = \{0\} \times T$

$W = \{[z,1] : |z| < \varepsilon\} \times T$ $\qquad$ $W^* = \{|z| < \varepsilon\} \times T$

$$f : W^* \setminus S^* \longrightarrow W \setminus S$$
$$(z, [\zeta]) \longmapsto \left([z,1], \left[\zeta + \frac{1}{2\pi i} \log z\right]\right)$$

clearly holomorphic, inverse $[\zeta] \longmapsto \left[\zeta - \frac{1}{2\pi i} \log z\right]$

well-definedness? $\quad z = re^{i\theta}$

$$\log z = \log r + i\theta$$

$$\zeta + \frac{1}{2\pi i} \log r + \frac{\theta}{2\pi} \quad \text{as} \quad \theta \longmapsto \theta + 2\pi n$$

It represents the same point on T.

2° visualizing it. For simplicity, take $\tau = i$

Topologically, glue two $D \times T^2$ along $S^1 \times T^2$

$M \setminus W \approx D \times T^2$ $\qquad$ $W^* \approx D \times T^2$

(w, α, β) $\qquad$ $(z, \tilde{\alpha}, \tilde{\beta})$

$\mathbb{CP}^1 \setminus \{[0,1]\}$
$= \mathbb{C} \approx D$ topologically

$\mathbb{R}/\mathbb{Z}$

Glue $S^1 \times T^2$ with $S^1 \times T^2$
by $(e^{i\theta}, \tilde{\alpha}, \tilde{\beta})$
$\sim (e^{-i\theta}, \tilde{\alpha} + \frac{\theta}{2\pi}, \tilde{\beta}) = (w, \alpha, \beta)$

$\Rightarrow \begin{cases} \beta = \tilde{\beta} \rightsquigarrow \text{an } S^1\text{-factor} \\ D \times S^1 \sim D \times S^1 \quad \Rightarrow \text{become } S^3 \\ (w, \alpha) \qquad (z, \tilde{\alpha}) \qquad (e^{-i\theta}, \alpha) \sim (e^{i\theta}, \alpha - \frac{\theta}{2\pi}) \end{cases}$

$\Rightarrow M^*$ is homeomorphic to $S^1 \times S^3$ $\boxed{\text{HW}}$

§V. blow-up (at a point)

1° $L_{-1} \longrightarrow \mathbb{CP}^1$ is the tautological bundle

(z, ξ) $\qquad$ $(w, \eta) = (z^{-1}, z\xi)$

$$\begin{tikzcd} L_{-1} \arrow[r] \arrow[rr, bend left, "f"] & \mathbb{CP}^1 \times \mathbb{C}^2 \arrow[r, "pr_2"] & \mathbb{C}^2 \end{tikzcd}$$

$$(z, \xi) \longmapsto ([z, 1], (z\xi, \xi))$$

$$(w, \eta) \longmapsto ([1, w], (\eta, w\eta))$$

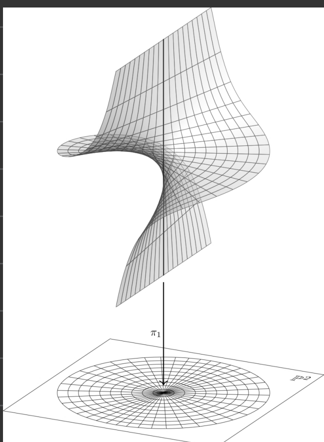
<u>observation</u> $\quad L_{-1} \setminus \{\mathbb{CP}^1 \text{ as the zero section}\} \overset{\text{biholomorphic}}{\cong} \mathbb{C}^2 \setminus \{0\}$

$\xi \neq 0, \ \eta \neq 0$

$0 < |z_0|^2 + |z_1|^2 < \varepsilon$ correspond to $0 < |\xi|^2 < (1+|z|^2)^{-1}\varepsilon$

$0 < |\eta|^2 < (1+|w|^2)^{-1}\varepsilon$

$2°$ M : a complex manifold of dimension 2



$S = p \in M$

W = a neighborhood, biholomorphic to $|z_0|^2 + |z_1|^2 < \varepsilon$ in $\mathbb{C}^2$

W^* = ε-neighborhood of the zero section in L_{-1}

S^* = zero section

gluing map f : as $1°$

$\Rightarrow$ M^* is called the blow-up of M at p.

The process removes p, and replace it by all directions in T_pM, which is a $\mathbb{CP}^1 = \mathbb{P}(\mathbb{C}^2)$

<u>remark</u> i) One can do this in higher dimensions

model : tautological bundle over $\mathbb{CP}^{n-1}$

ii) This can be used to resolve singularities.

§ VI. recall : some linear algebra

1° Hermitian inner product on $V = \mathbb{C}^n$

$\langle u, v\rangle = \overline{\langle v, u\rangle}$

$\langle u, u\rangle \geq 0$ "=" only when $u=0$

$\langle \lambda u, v\rangle = \lambda \langle u, v\rangle$, $\langle u, \lambda v\rangle = \bar{\lambda}\langle u, v\rangle$

$\rightsquigarrow \langle u, v\rangle = v^* H u$, H: Hermitian, with positive eigenvalues

2° inner product on $\mathbb{R}^{2n} \cong \mathbb{R}^n \oplus i\,\mathbb{R}^n = \mathbb{C}^n$

$u = x_1 + i y_1$ $\quad v = x_2 + i y_2$

$H = G + iW$ $\qquad H^* = G^T - i W^T$

$\Rightarrow$ G is symmetric. W is skew-symmetric

$$\begin{aligned}\langle u, v\rangle &= (x_2^T - i y_2^T)(G + iW)(x_1 + i y_1) \\ &= (x_2^T G x_1 + y_2^T G y_1 - x_2^T W y_1 + y_2^T W x_1) \rightsquigarrow \begin{bmatrix} G & -W \\ W & G \end{bmatrix} \\ &\quad + i\,(x_2^T W x_1 + y_2^T W y_1 + x_2^T G y_1 - y_2^T G x_1) \rightsquigarrow \begin{bmatrix} W & G \\ -G & W \end{bmatrix}\end{aligned}$$

Since $\mathrm{Re}\langle u, v\rangle$ is an inner product, g. (over $\mathbb{R}$)

$\begin{bmatrix} G & -W \\ W & G \end{bmatrix}$ is symmetric, positive definite

$\mathrm{Im}\langle u, v\rangle$ corresponds to $\begin{bmatrix} W & G \\ -G & W \end{bmatrix}$, which gives a skew-symmetric bilinear form on $\mathbb{R}^{2n}$.

3° $J(x, y) = (-y, x)$ rotate couterclockwise by $\pi/2$

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} G & -W \\ W & G \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -W & -G \\ G & -W \end{bmatrix}$$

$\Rightarrow$ with J and g, we can reconstruct H.