1° $W \overset{\text{open}}{\subset} \mathbb{C}^n$, $f : W \to \mathbb{C}$ is holomorphic if for any $(a_1, \cdots, a_n) \in W$

$$f(z) = \sum_{k_j \geq 0} C_{k_1 \cdots k_n} (z_1 - a_1)^{k_1} \cdots (z_n - a_n)^{k_n}$$ is some neighborhood of a

<u>theorem</u> (Osgood) f : continuous on W, holomorphic with respect to each z_k when others are fixed. Then f is holomorphic

pf: <u>poly disk</u> $P(a, r) = \{ z : |z_j - a_j| < r_j \quad j = 1, \cdots, n \}$

$r = (r_1, \cdots, r_n)$

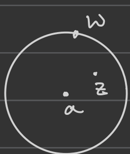
For any $a \in W$, choose $\overline{P(a,r)} \subset W$.

Apply Cauchy integral formula for each variable

For $z \in P(a,r)$ $$f(z_1, \cdots, z_n) = \frac{1}{2\pi i} \int_{|w_1 - a_1| = r_1} \frac{f(w_1, z_2, \cdots, z_n)}{w_1 - z_1} \, dw_1$$

and then $$f(w_1, z_2, \cdots, z_n) = \frac{1}{2\pi i} \int_{|w_2 - a_2| = r_2} \frac{f(w_1, w_2, z_3, \cdots, z_n)}{w_2 - z_2} \, dw_2$$

$$\cdots \Rightarrow f(z_1, \cdots, z_n) = \left(\frac{1}{2\pi i}\right)^n \int \cdots \int_{|w_j - a_j| = r_j} \frac{f(w_1, \cdots, w_n)}{(w_1 - z_1) \cdots (w_n - z_n)} \, dw_1 \cdots dw_n$$



$$\left| \frac{z_j - a_j}{w_j - a_j} \right| < 1 \Rightarrow \frac{1}{w_j - z_j} = \frac{1}{(w_j - a_j) - (z_j - a_j)}$$

$$= \frac{1}{w_j - a_j} \sum_{k_j = 0}^{\infty} \left(\frac{z_j - a_j}{w_j - a_j} \right)^{k_j}$$

not integrated

$$\Rightarrow f(z_1, \cdots, z_n) = \sum_{k_j \geq 0} \left(\underbrace{\left(\frac{1}{2\pi i}\right)^n \int \cdots \int_{|w_j - a_j| = r_j} \frac{f(w_1, \cdots, w_n)}{(w_1 - a_1)^{k_1 + 1} \cdots (w_n - a_n)^{k_n + 1}} \, dw_1 \cdots dw_n}_{\leq M r_1^{-k_1} \cdots r_n^{-k_n}} \right) (z_1 - a_1)^{k_1} \cdots (z_n - a_n)^{k_n}$$

$\Rightarrow$ convergent ※

<u>notation</u> As the 1-dimensional case.

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right)$$

$$\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right)$$

holomorphic $\Leftrightarrow$ $\frac{\partial f}{\partial \bar{z}_j} = 0 \quad \forall j$

2° $f = (f_1, \cdots, f_m) : \mathcal{U} \subset \mathbb{C}^n \to \mathbb{C}^m$

Suppose that f is holomorphic, i.e. each f_j is, the Jacobian matrix is
$$\begin{bmatrix} \frac{\partial f_1}{\partial z_1} & & \frac{\partial f_1}{\partial z_n} \\ & \ddots & \\ \frac{\partial f_m}{\partial z_1} & & \frac{\partial f_m}{\partial z_n} \end{bmatrix} = \left(\frac{\partial f_j}{\partial z_k} \right)$$

When $m=n$, write $f_j = u_j + i v_j$, $z_j = x_j + i y_j$

consider the $2n \times 2n$ Jacobian, denote its determinant by
$$\frac{\partial(u,v)}{\partial(x,y)} = \det \left[\frac{\partial(u_1, v_1, \cdots, u_n, v_n)}{\partial(x_1, y_1, \cdots, x_n, y_n)} \right]$$

<u>claim</u> If f is holomorphic, $\frac{\partial(u,v)}{\partial(x,y)} = \left| \det\left(\frac{\partial f_j}{\partial z_k} \right) \right|^2$

pf for $n=2$: $a_{jk} = \frac{\partial u_j}{\partial x_k} = \frac{\partial v_j}{\partial y_k}$, $b_{jk} = \frac{\partial u_j}{\partial y_k} = -\frac{\partial v_j}{\partial x_k}$ Cauchy-Riemann

$$\frac{\partial f_j}{\partial z_k} = \frac{1}{2}\left(\frac{\partial(u_j + i v_j)}{\partial x_k} - i \frac{\partial(u_j + i v_j)}{\partial y_k} \right) = a_{jk} - i b_{jk}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} a_{11} & b_{11} & a_{12} & b_{12} \\ -b_{11} & a_{11} & -b_{12} & a_{12} \\ a_{21} & b_{21} & a_{22} & b_{22} \\ -b_{21} & a_{21} & -b_{22} & a_{22} \end{vmatrix}$$

do row/column operation over $\mathbb{C}$

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \rightsquigarrow \begin{bmatrix} a-ib & i(a-ib) \\ -b & a \end{bmatrix} \rightsquigarrow \begin{bmatrix} a-ib & 0 \\ -b & a+ib \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} \partial f_1/\partial z_1 & 0 & \partial f_1/\partial z_2 & 0 \\ * & \overline{(\partial f_1/\partial z_1)} & * & \overline{(\partial f_1/\partial z_2)} \\ \partial f_2/\partial z_1 & 0 & \partial f_2/\partial z_2 & 0 \\ * & \overline{(\partial f_2/\partial z_1)} & * & \overline{(\partial f_2/\partial z_2)} \end{bmatrix} \Rightarrow \text{Done} \quad ※$$

<u>theorem</u> (IFT) $f : \mathcal{U} \subset \mathbb{C}^n \to \mathbb{C}^n$ holomorphic. If $\det\left(\frac{\partial f_j}{\partial z_k} \right)\Big|_a \neq 0$

Then $\exists$ neighborhood N of a such that $f : N \to f(N)$ is a diffeomorphism with f^{-1} being holomorphic

pf: Above claim $\Rightarrow$ apply IFT on $\mathbb{R}^{2n}$.

It remains to check that f^{-1} is holomorphic

$$z_j \underset{\varphi}{\overset{f}{\rightleftarrows}} \mathbb{C}^n \cong \mathbb{R}^{2n}$$

We know $\varphi = \varphi(u_1, v_1, \cdots, u_n, v_n)$

$= \varphi(w_1, \bar{w}_1, \cdots, w_n, \bar{w}_n)$

$$0 = \frac{\partial z_j}{\partial \bar{z}_k} = \frac{\partial \varphi_j}{\partial w_\ell}\cancel{\frac{\partial f_\ell}{\partial \bar{z}_k}}^{0} + \frac{\partial \varphi_j}{\partial \bar{w}_\ell}\frac{\partial \bar{f}_\ell}{\partial \bar{z}_k} \Rightarrow \frac{\partial \varphi_j}{\partial \bar{w}_\ell} = 0$$

invertible

3° complex manifold <u>defn</u> model = open subset of $\mathbb{C}^n$

transition = biholomorphic

<u>remark</u> same to use open ball in $\mathbb{C}^n$ (by shrinking)

but not whole $\mathbb{C}^n$.

unlike smooth manifold : $\mathbb{R}^n \cong B(0;1)$ diffeomophic.

<u>basic example</u> : $\mathbb{CP}^n = \{$ comple lines in $\mathbb{C}^{n+1}\} = \mathbb{P}(\mathbb{C}^{n+1})$

$= \mathbb{C}^{n+1} \setminus \{0\} \big/ \mathbb{C}^*$

$(Z_0, Z_1, \cdots, Z_n) = (W_0, W_1, \cdots, W_n]$ iff $Z_j = \alpha W_j$ $j=0,1,\cdots,n$ for some $\alpha \in \mathbb{C}^*$

coordinate chart : Suppose $Z_0 \neq 0$

$(Z_0, Z_1, \cdots, Z_n) \sim (1, Z_1/Z_0, \cdots Z_n/Z_0)$

$\mathbb{C}^n$ $\mathbb{CP}^n$

$\rightsquigarrow$ $(z_1, \cdots, z_n) \longmapsto [(1, z_1, \cdots, z_n)]$ provides a chart

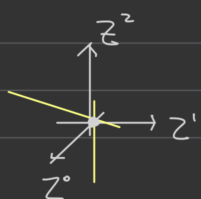
another one $(w_1, \cdots, w_n) \longmapsto [(w_1, 1, w_2, \cdots, w_n)]$

overlap region : $w_1 \neq 0$, $z_1 \neq 0$

transition ? $(1, z_1, \cdots, z_n) \sim (z_1^{-1}, 1, z_1^{-1}z_2, \cdots, z_1^{-1}z_n)$

$\Rightarrow$ $w_1 = z_1^{-1}$, $w_j = z_1^{-1} z_j$ $j = 2, 3, \cdots n$

clearly holomorphic



Unless the line $\subset Z^1 \cdots Z^n$ hyperplane

Otherwise it is a graph over Z^0. parametrized by slope

This is the meaning of the chart