§0. hairy ball theorem

<u>theorem</u> S^{2n} does not admits a nowhere vanishing tangent vector field.

$\rightsquigarrow$ Study sections of $\mathbb{R}^{2n} \longrightarrow TS^{2n} \xrightarrow{\pi} S^{2n}$

In general, consider $\mathbb{R}^n \longrightarrow E \xrightarrow{\pi} M^n$

<u>remark</u> dimension of base = rank of the bundle

local sections : $U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ generically has discrete zeros.

For the Pontryagin class, $P_j(E) \in H^{4j}_{dR}(M)$ only to $j = [n/4]$

§1. some linear algebra : $\sqrt{\det}$

1° $O(n) = \{ A \in Gl(n;\mathbb{R}) : A^t A = \mathbb{1} \}$

$= SO(n) \amalg SO(n) \times \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$

$\hookrightarrow \det A = 1$

2° (For metric connections, curvature is a skew-symmetric matrix of 2-forms. $\det(F^\nabla) \in H^{2n}_{dR}(M;\mathbb{R}) = 0$ if $\dim M = n$)

Consider the determinant of a skew-symmetric matrix

n	2	3	4
	$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}$
det	a^2	$a\begin{vmatrix} a & -b \\ -c & 0 \end{vmatrix} + b\begin{vmatrix} a & -b \\ 0 & c \end{vmatrix}$ $= -abc + abc = 0$	$(af - be + cd)^2$

<u>observation</u> n : even $\Rightarrow$ det has a square root

n : odd $\Rightarrow$ det is zero. $\boxed{HW}$

3° invariant polynomial ?

If $X = -X^T$ and $A \in Gl$, $(A^{-1}XA)^T = -A^T X (A^{-1})^T$ needs not to be skew-symmetric

When $A^T = A^{-1}$ $(A \in O(n))$, it is still skew-symmetric

Consider $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $X = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$ not invariant

$$A^{-1}XA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}$$

4° <u>defn</u> The Pfaffian of skew-symmetric matrix X is

$$Pf(X) = \frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} sgn(\sigma) X_{\sigma_1 \sigma_2} X_{\sigma_3 \sigma_4} \cdots X_{\sigma_{2m-1} \sigma_{2m}}$$

when X is of size $2m \times 2m$. It is defined to be zero if X is of size $(2m-1) \times (2m-1)$

<u>prop</u> i) $(Pf(X))^2 = \det(X)$

ii) $Pf(AXA^{-1}) = \det(A) \cdot Pf(X)$ $\forall A \in O(2m)$

Pf: for ii) $AXA^{-1} = AXA^T$

$$(AXA^{-1})_{\sigma_1 \sigma_2} = \sum_{\tau_1, \tau_2} A_{\sigma_1 \tau_1} X_{\tau_1 \tau_2} A_{\tau_2 \sigma_2}$$

$2^m m! \, Pf(AXA^{-1})$

$$= \sum sgn(\sigma) (A_{\sigma_1 \tau_1} X_{\tau_1 \tau_2} A_{\sigma_2 \tau_2}) \cdots (A_{\sigma_{2m-1} \tau_{2m-1}} X_{\tau_{2m-1} \tau_{2m}} A_{\sigma_{2m} \tau_{2m}})$$

$$= \sum \underbrace{sgn(\tau) X_{\tau_1 \tau_2} \cdots X_{\tau_{2m-1} \tau_{2m}}}_{Pf(X)} \cdot \underbrace{sgn(\gamma = \sigma^{-1}\tau) A_{1 \gamma_1} \cdots A_{2m \gamma_{2m}}}_{\det(A)}$$

for i) For $X^* = -X$, one can find $A \in O(n = 2m)$ such that

$$AXA^{-1} = \begin{bmatrix} 0 & a_1 & & & \\ -a_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & a_m \\ & & & -a_m & 0 \end{bmatrix}$$

by ii), $\pm Pf(X) = a_1 \cdots a_m$, it is also clear that

$\det(X) = (a_1 \cdots a_m)^2$ ※

§ III. Euler class

1°. <u>defn</u> a real vector bundle is said to be orientable if $\exists \{U_\alpha\}_\alpha$ with $g_{\alpha\beta} = U_\alpha \cap U_\beta \longrightarrow \underset{\det > 0}{Gl^+(n;\mathbb{R})}$

<u>defn</u> Suppose that E is orientable. Choose a bundle metric and a metric connection ∇. If we use oriented, orthonormal trivializations, the local expression of F^∇ differs by the conjugation of $SO(\text{rank } E)$

If $\text{rank } E = 2m+1$, the Euler form of F^∇ is defined to be zero. If $\text{rank } E = 2m$, the Euler form is defined to be $eu(F^\nabla) = \frac{1}{(2\pi)^m} Pf(F^\nabla) \in \Omega^{2m}(M;\mathbb{R})$

2° <u>prop</u> i) $eu(F^\nabla)$ is d-closed.

ii) $[eu(F^\nabla)] \in H^{2m}_{dR}(M)$ only depends on E and its orientation, but not the metric and connection

$\rightsquigarrow$ $eu(E)$: the Euler class of E

- $eu(E) \wedge eu(E) = P_m(E) \in H^{4m}_{dR}(M)$
- Given a bundle metric, $\langle\ ,\ \rangle$, and a connection, ∇ one can construct a metric connection, $\nabla^{\natural}$ as follows
  $$\mathfrak{a}(X, s_1, s_2) = X(\langle s_1, s_2\rangle) - \langle \nabla_X s_1, s_2\rangle - \langle s_1, \nabla_X s_2\rangle$$
  It is straightforward to check that $\mathfrak{a}$ is $\mathcal{C}^\infty(M;\mathbb{R})$ linear in all three arguments
  $\Rightarrow$ Define $\langle \nabla^{\natural}_X s_1, s_2\rangle$ to be $\langle \nabla_X s_1, s_2\rangle + \frac{1}{2}\mathfrak{a}(X, s_1, s_2)$
- The proof of the proposition is pretty much the same as that for the Pontryagin / Chern classes.
  (g_0, ∇^0) , (g_1, ∇^1)
  $\rightsquigarrow$ $g_t = (1-t) g_0 + t g_1$, $\nabla_t = ((1-t)\nabla^0 + t\nabla^1)^{\natural}$ ...

3° <u>theorem</u> For an oriented $\mathbb{R}^{2m}$ vector bundle $E \xrightarrow{\pi} M$, if it admits a nowhere vanishing section, then $eu(E) = 0$ (in $H^{2m}_{dR}(M)$)

pf: $s \in \Gamma(E)$, s: nowhere zero.

Choose a bundle metric on $E \Rightarrow E \cong \mathbb{R}\langle s\rangle \oplus \langle s\rangle^{\perp}$

oriented $(2m-1)$-bundle

Define ∇ on $\mathbb{R}\langle s\rangle$ by $\nabla \frac{s}{|s|} = 0$

on $\langle s\rangle^{\perp}$: any metric connection $\Rightarrow eu(F^{\nabla}) = 0$

※

<u>remark</u>
- If M^n is compact, oriented, the homomorphism
$$H^n_{dR}(M) \longrightarrow \mathbb{R}$$
$$\omega \longmapsto \int_M \omega$$
is actually an isomorphism
- When $\operatorname{rank} E = 2m = \dim M$, it is useful to look at $\int_M eu(E)$ $(\in \mathbb{Z})$

4° (review Gauss–Bonnet) For Σ^2: compact, oriented, without boundary, endow a metric g

$\rightsquigarrow \nabla =$ Levi-Civita

$K_g =$ Gauss curvature $= \langle R(e_1, e_2) e_2, e_1 \rangle$ for $e_1 e_2$: orthonormal

Suppose that $\{e_1, e_2\}$: local, oriented, orthonormal frame for $T\Sigma$. Denote by $\{\omega^1, \omega^2\}$ the dual basis

$\Rightarrow \omega^1 \wedge \omega^2 > 0$, $\int_\Sigma \omega^1 \wedge \omega^2 = \operatorname{Vol}(\Sigma, g)$

Since $\Lambda^2 T^* \Sigma = \mathbb{R}\langle \omega^1 \wedge \omega^2 \rangle$. $F^{\nabla} = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \omega^1 \wedge \omega^2$

where $a = \langle F^{\nabla}(e_1, e_2) e_2, e_1 \rangle = K_g$

$$\Rightarrow \quad F^\nabla = \begin{bmatrix} 0 & K_g \\ -K_g & 0 \end{bmatrix} \omega^1\wedge\omega^2 \quad \Rightarrow \quad eu(T\Sigma) = \frac{1}{2\pi} K_g\, \omega^1\wedge\omega^2$$

By Gauss-Bonnet $\displaystyle\int_\Sigma eu(T\Sigma) = \frac{1}{2\pi}\int K_g\,\omega^1\wedge\omega^2$

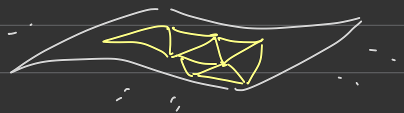
$$= \chi(\Sigma) = 2 - 2\,\text{genus}(\Sigma)$$

<u>remark</u> For S^2, $eu(TS^2) \neq 0$.

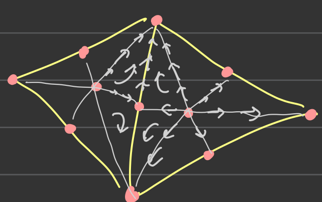
$\Rightarrow$ admits no nowhere vanishing tangent vector field

5° (revisiting 4°) $\displaystyle\int_\Sigma eu(T\Sigma) = \chi(\Sigma)$ without invoking Gauss-Bonnet?

i) Choose a "triangulation" of Σ

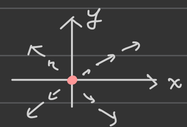


ii) Construct a vector field V as follows:



zeros	of V
vertex	sink
edge	saddle
face	source

iii) Near vertex



$$V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

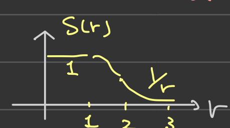
$$g_{std} = dx^2 + dy^2$$

<u>goal</u> Construct g and ∇ such that

$|V| = 1$ and $\nabla V = 0$ on the annulus region

needs not to be Levi-Civita

Choose $g = (s(r))^2\left((dx)^2 + (dy)^2\right)$



$\Rightarrow$ $S^{-1}\frac{\partial}{\partial x}$, $S^{-1}\frac{\partial}{\partial y}$: oriented, orthonormal basis

With respect to it, $\nabla = d + \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}$ α : smooth 1-form

Requirement $\nabla\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) = 0$ on $B(3)\setminus B(2)$

$$= \nabla\left(\frac{x}{r}\, r\frac{\partial}{\partial x} + \frac{y}{r}\, r\frac{\partial}{\partial y}\right)$$

$$= d\left(\frac{x}{r}\right) r\frac{\partial}{\partial x} - \frac{x}{r}\,\alpha\, r\frac{\partial}{\partial y} + d\left(\frac{y}{r}\right) r\frac{\partial}{\partial y} + \frac{y}{r}\,\alpha\, r\frac{\partial}{\partial x}$$

$\left(dr = \frac{x\,dx + y\,dy}{r}\right)$ $\Rightarrow$ $\begin{cases} \frac{x}{r}\alpha = d\left(\frac{y}{r}\right) \\ \frac{y}{r}\alpha = -d\left(\frac{x}{r}\right) \end{cases}$ $\Rightarrow$ $\alpha = \frac{-y\,dx + x\,dy}{r^2} = d\theta$

note that $d\alpha = 0$

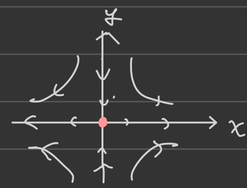
Choose $\alpha = S^2(-y\,dx + x\,dy) = S^2 r^2\, d\theta$ $(S \cdot r = 1$ on $B(3)\setminus B(2))$

$F = \begin{bmatrix} 0 & d\alpha \\ -d\alpha & 0 \end{bmatrix}$ $\Rightarrow$ $\frac{1}{2\pi}\mathrm{Pf}(F) = \frac{1}{2\pi}d\alpha$: only nonzero on $B(2)$

$$\iint_{B(2)} \frac{1}{2\pi}\mathrm{Pf}(F) = \frac{1}{2\pi}\int_{\partial B(2)} d\theta = 1$$

iv) Near the barycenter of the edge

$V = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$



Similarly, choose $g = (S(r))^2\left((dx)^2 + (dy)^2\right)$

$0 = \nabla V = \nabla\left(\frac{x}{r}\, r\frac{\partial}{\partial x} - \frac{y}{r}\, r\frac{\partial}{\partial y}\right)$ on $B(3)\setminus B(2)$

$$= d\left(\frac{x}{r}\right) r\frac{\partial}{\partial x} - \frac{x}{r}\,\alpha\, r\frac{\partial}{\partial y} - d\left(\frac{y}{r}\right) r\frac{\partial}{\partial y} - \frac{y}{r}\,\alpha\, r\frac{\partial}{\partial x}$$

$\begin{cases} \frac{x}{r}\alpha = -d\left(\frac{y}{r}\right) \\ \frac{y}{r}\alpha = +d\left(\frac{x}{r}\right) \end{cases}$ $\Rightarrow$ $\alpha = \frac{+y\,dx - x\,dy}{r^2} = -d\theta$

Parallel argument $\Rightarrow$ $\frac{1}{2\pi}\mathrm{Pf}(F) = \frac{1}{2\pi}d\alpha$: only nonzero on $B(2)$

$$\iint_{B(2)} \frac{1}{2\pi}\mathrm{Pf}(F) = \frac{1}{2\pi}\int_{\partial B(2)} -d\theta = -1$$

v) Near the barycenter of the face : similar to iii)

vi) Finally, extend g and ∇ to $\Sigma \setminus \left\{ \begin{array}{c} \text{neighborhood of} \\ \text{zeros of } V \end{array} \right\}$

such that $|V| = 1$ and $\nabla V = 0$ there

$$\Rightarrow \int_\Sigma eu(T\Sigma) = \#\{\text{vertices}\} - \#\{\text{edges}\} + \#\{\text{faces}\}$$

<u>remark</u> In general, for M^{2m}, compact, oriented, boundaryless

$$\int_M eu(T\Sigma) = \chi(M)$$

See S.-S. Chern. Ann. of Math. 1944