§I. curvature.

1° ∇ : connection on $E \xrightarrow{\pi} M$

$\rightsquigarrow F^{\nabla} \in \Gamma(\Lambda^2 T^*M \otimes \operatorname{End} E) = \Omega^2(\operatorname{End} E)$

endomorphism valued 2-form

key feature: the curvature is not a differential operator

goal construct "invariant" of E from F^{∇}

keep in mind that ∇ is NOT unique

2° With a local trivialization, $\nabla = d + A$. $F_A = dA + A \wedge A$

For different choices of trivializations,

$$\tilde{A} = g^{-1}dg + g^{-1}Ag \Rightarrow F_{\tilde{A}} = g^{-1}F_A\, g$$

3° Consider $Gl(n;\mathbb{R}) \curvearrowright M(n \times n;\mathbb{R})$ by conjugation

Find some functions on $M(n\times n;\mathbb{R})$ which is invariant under the conjugation action

example $T \in M(n \times n;\mathbb{R})$, $\operatorname{tr} T$ and $\det T$ are invariant

In general. consider $\det(\mathbb{I} + sT)$

$$= 1 + s f_1(T) + \cdots + s^n f_n(T)$$

Each $f_j(T)$ is invariant, and is a degree k polynomial in the entries of T

$\boxed{\text{check}}$ $f_1(T) = \operatorname{tr}(T)$, $f_n(T) = \det(T)$

§II. characteristic class (or $\mathbb{C}^n$)

1° E : vector bundle $\mathbb{R}^n \longrightarrow E \xrightarrow{\pi} M$, with connection ∇

Consider $\det(\mathbb{I} + sF^{\nabla}) = 1 + s f_1(F^{\nabla}) + \cdots + s^n f_n(F^{\nabla})$

Note $f_j(-)$ is a degree j polynomial in the entries of $(-)$.

$\Rightarrow f_j(F^{\nabla})$ is a $2j$-form on M

($\wedge$ on $\bigoplus_{j\geq 0} \Omega^{2j}(M)$ is abelian)

2° At this moment $f_j(F^\nabla) \in \Omega^{2j}(M;\mathbb{R})$ for real ones

$\Omega^{2j}(M;\mathbb{C})$ for complex ones

prop 1 $f_j(F^\nabla)$ is d-closed.

pf: Choose a local trivialization, and $\nabla = d + A$

$$\det(\mathbb{I} + sF_A) = 1 + \sum_{j\ge 1} s^j \underbrace{f_j(F_A)}_{2j\text{-forms}}$$

$$d(\det(\mathbb{I} + sF)) = \det(\mathbb{I} + F) \wedge \operatorname{tr}((\mathbb{I}+sF)^{-1} \wedge d(\mathbb{I}+sF))$$

By Cramer's rule

where $(\mathbb{I}+sF)^{-1} = \mathbb{I} - sF + s^2F^2 - \cdots \pm (-1)^j s^j F^j + \cdots$

indeed a finite sume

$$d(\mathbb{I}+sF) = s\, dF$$

$$= s(F\wedge A - A\wedge F) \text{ by the Bianchi identity}$$

$$\Rightarrow \operatorname{tr}((\mathbb{I}+sF)^{-1} d(\mathbb{I}+sF))$$

$$= s \operatorname{tr}((\mathbb{I} - sF + s^2F \cdots) \wedge (F\wedge A - A\wedge F)$$

$$= \sum_{j\ge 0} s^{j+1}(-1)^j \underbrace{\operatorname{tr}(F^j \wedge (F\wedge A - A \wedge F))}$$

$$= \operatorname{tr}(F^{j+1}\wedge A) - \operatorname{tr}(A\wedge F^{j+1}) = 0$$

3° different choices of ∇ ? ※

preliminary. $\varphi : X \to M$, $E \xrightarrow{\pi} M$ with ∇.

It naturally gives a connection $\varphi^*\nabla$ on φ^*M

Suppose that $\{U_\alpha\}_\alpha$: open cover of M

with $E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^n$. $\Rightarrow$ ∇ on $U_\alpha = d + A_\alpha$

⇝ φ^*E is given by the open cover $\{\tilde{U}_\alpha = \varphi^{-1}(U_\alpha)\}_\alpha$

with the transition $\tilde{g}_{\alpha\beta} = g_{\alpha\beta}\circ\varphi : \tilde{U}_\alpha \cap \tilde{U}_\beta \longrightarrow Gl(n;\mathbb{R})$

Define $\varphi^*\nabla$ by $d + \varphi^*A_\alpha$ on each $\tilde{U}_\alpha$

↪ $n\times n$-matrix of 1-forms on $\tilde{U}_\alpha$

$$\varphi^*A_\beta = g^{-1}dg + g^{-1}\cdot\varphi^*A_\alpha\cdot g \quad \text{for } g = \tilde{g}_{\alpha\beta}$$

$\boxed{\text{check}}$ chain rule / change of variable

from $A_\beta = g^{-1}dg + g^{-1}A_\alpha g$ for $g = g_{\alpha\beta}$

prop 2 As a class in $H^{2j}_{dR}(M)$, $f_j(F^{\nabla})$ is well-defined.
Namely, $[f_j(F^{\nabla})] \in H^{2j}_{dR}(M)$ is independent of the choice of connections.

pf: i) Suppose that ∇^0 and ∇^1 are connections on E

Consider
$$\pi : M \times \mathbb{R} \to M, \quad (p, t) \mapsto p$$

Both $\pi^*\nabla^0$ and $\pi^*\nabla^1$ are connections on π^*E

$\Rightarrow \tilde{\nabla} = (1-t)\,\pi^*\nabla^0 + t\,\pi^*\nabla^1$ is a connection on π^*E

ii) For $E|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R}^n$ $\quad \nabla^0 = d + A_0$, $\nabla^1 = d + A_1$

only in x

$\Rightarrow \tilde{\nabla} = d + (1-t)A_0 + t A_1$

on $M \times \mathbb{R}$; also in t

$$\tilde{F} = d((1-t)A_0 + tA_1) + ((1-t)A_0 + tA_1) \wedge ((1-t)A_0 + tA_1)$$
$$= dt \wedge (A_1 - A_0) + (1-t)F_{A_0} + tF_{A_1} + t(1-t)(\cdots)$$
$$\tilde{F}|_{t=0} = F_{A_0}, \qquad \tilde{F}|_{t=1} = F_{A_1}$$

iii)
$$f_j(\tilde{F}) = dt \wedge \underset{\Omega^{2j-1}(M)}{\underset{\cap}{a_j(t)}} + \underset{\in\, \Omega^{2j}(M)}{b_j(t)}$$

Note that $f_j(-)$ is algebraic, and hence $f_j(\tilde{F})|_{t=0} = f_j(\tilde{F}|_{t=0})$

$\Rightarrow b_j(0) = f_j(F_{A_0})$ $\quad b_j(1) = f_j(F_{A_1})$

Consider
$$\iota_t : M \to M \times \mathbb{R}, \quad p \mapsto (p, t)$$
ι_0 is homotopic to ι_1

$$\Rightarrow \underset{[f_j(F_{A_0})]}{\underset{\|}{[\iota_0^* f_j(\tilde{F})]}} = \underset{[f_j(F_{A_1})]}{\underset{\|}{[\iota_1^* f_j(\tilde{F})]}} \quad \text{in } H^{2j}_{dR}(M)$$

※

4° **remark** For $\varphi : X \to M$ with $E \xrightarrow{\pi} M$,
$$f_j(F_{\varphi^*\nabla}) = \varphi^*(f_j(F_\nabla))$$
(Since $F_{\varphi^*\nabla} = \varphi^* F_\nabla$, f_j is a polynomial in components of F)

§ III. Pontryagin class

1° Chose a bundle metric on the real vector bundle E.

<u>claim</u> There exists a metric connection ∇ on E

Namely, $d\langle s_1, s_2\rangle = \langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle$

for any two sections s_1, s_2

pf: Locally, $E|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R}^n$. we may assume the inner product is the standard one on $\mathbb{R}^n$

(It is equivalent to find local trivializing sections which are everywhere orthonormal. One can start with $\{\tilde{s}_1, \cdots, \tilde{s}_n\}$, then do (pointwise) Gram-Schmidt process)

Then, d induces a metric connection on $E|_{\mathcal{U}}$.

$\Rightarrow$ Same partition of unity construction does the job ※

2° Now, suppose ∇ is a metric connection. what can be said about A?

Again, assume $E|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R}^n$, and the metric is standard.

Consider (constant) orthonormal sections. $e_1, \cdots, e_n$

The (i,j)-component of A is $\langle \nabla e_j, e_i\rangle$

$$\left(\begin{aligned} \nabla(s^j e_j) &= (ds^j)\, e_j + s^j \nabla e_j \\ &= (ds^i) e_i + \langle \nabla e_j, e_i\rangle s^j e_i \end{aligned}\right)$$

Since $\langle e_j, e_i\rangle = \delta_{ji}$, $\langle \nabla e_j, e_i\rangle + \langle e_j, \nabla e_i\rangle = 0$

<u>upshot</u> In terms of orthonormal trivialization, A is a skew-symmetric matrix of 1-forms.

$$\Rightarrow F_A = dA + A \wedge A$$

$$F_A^* = dA^* \underset{\substack{\uparrow \\ A\,:\,1\text{-forms}}}{-} A^* \wedge A^* = -dA - A\wedge A = -F_A$$

In particular, $\mathrm{tr}(F_A) = \mathrm{tr}(F_A^*) = -\mathrm{tr}(F_A) \Rightarrow \mathrm{tr}(F_A) = 0$

$\Rightarrow [f_1(E)] = 0$ in $H^2_{dR}(M)$

3° In general, if $F_A^* = -F_A$

$$\det(\mathbb{1} + sF_A) = 1 + \sum_{j\geq 1} s^j f_j(F_A) \qquad \Rightarrow f_{2i+1}(F_A) = 0$$
$$= \det(\mathbb{1} + sF_A^*) = 1 + \sum_{j\geq 1} (-1)^j s^j f_j(F_A)$$

defn $\left[\det\left(\mathbb{1} + \frac{1}{2\pi} F_A\right)\right] = 1 + P_1(E) + \cdots + P_{n/2}(E) \in \bigoplus_{j\geq 0} H^{4j}_{dR}(M)$

is called the (total) Pontryagin class.

($P_j(E) \in H^{4j}_{dR}(M; \mathbb{Z})$ → in algebraic topology)

§ IV. Chern class

1° Now, focus on complex vector bundles. Everything is required to be $\mathbb{C}$-linear

defn ∇: a connection on E

$$\left[\det\left(\mathbb{1} + \frac{i}{2\pi} F^\nabla\right)\right] = 1 + \sum_{j\geq 1} c_j(E) \in \bigoplus_{j\geq 0} H^{2j}_{dR}(M; \mathbb{C})$$

is called the (total) Chern class of E

2° **claim** $c_j(E)$ is real, $c_j(E) \in H^{2j}_{dR}(M; \mathbb{R})$

pf: Again, endow E a (Hermitian) bundle metric, and a metric connection

$E|_U = U \times \mathbb{C}^n$. Consider the unitary sections $\{e_j\}_{j=1}^n$

$$\nabla(s^j e_j) = (ds^j) e_j + s^j \nabla e_j$$
$$= (ds^i) e_i + \underbrace{\langle \nabla e_j, e_i \rangle}_{(i,j)\text{-component of } A} s^j e_i$$

$$0 = d\langle e_j, e_i \rangle = \langle \nabla e_j, e_i \rangle + \langle e_j, \nabla e_i \rangle \Rightarrow A_{ij} = -\bar{A}_{ji}$$

$\Rightarrow$ A is skew-Hermitian in terms of unitary trivializations

$$F_A = dA + A \wedge A$$
$$F_A^* = dA^* - A^* \wedge A^* = -d\bar{A} - \bar{A} \wedge \bar{A} = -\overline{F_A}$$

$$1 + \sum_{j\geq 1} c_j(F_A) = \det\left(\mathbb{1} + \frac{i}{2\pi} F_A\right) = \det\left(\mathbb{1} + \frac{i}{2\pi} F_A^t\right)$$
$$= \overline{\det\left(\mathbb{1} + \frac{i}{2\pi} F_A\right)} = 1 + \sum_{j\geq 1} \overline{c_j(F_A)}$$

※

§V. examples. (over $\mathbb{C}$)

0° If $E_1 \overset{\Phi}{\cong} E_2$, then $c_j(E_1) = c_j(E_2)$ for all j

$$\begin{tikzcd} E_1 \arrow[dr] & & E_2 \arrow[dl] \\ & M & \end{tikzcd}$$

pf: Choose a connection ∇ on E_2

Consider $\varphi^*\nabla$ on E_1

Locally. $\varphi \in Gl(n;\mathbb{C})$. $\nabla = d + A$

$$(\varphi^*\nabla)s = \varphi^{-1}(\nabla(\varphi s)) = \varphi^{-1}(d+A)(\varphi s)$$

$$= (d + \varphi^{-1}A\varphi + \varphi^{-1}d\varphi)(s)$$

$$\Rightarrow \varphi^*\nabla = d + \varphi^{-1}A\varphi + \varphi^{-1}d\varphi$$

$$\Rightarrow F^{\varphi^*\nabla} = \varphi^{-1}F^{\nabla}\varphi. \quad ※$$

1° On $S^2 = \mathbb{C} \cup \{\infty\} = \underset{z}{\mathbb{C}}^{=:U} \cup \underset{w}{\mathbb{C}}^{=:V}$

For any $n \in \mathbb{Z}$. define L_n by $g_{VU} = w^n = z^{-n}$

i) Consider $n = -1$. <u>claim</u> L_{-1} is the tautological line bundle

$$(z, \xi) \sim (w, \eta)$$

$$w = \frac{1}{z},\ \eta = \frac{\xi}{w} = z\xi$$

$$L_{-1} \longrightarrow \mathbb{C}P^1 \times \mathbb{C}^2$$

$$(z,\xi) \longmapsto ([z,1], (\xi z, \xi))$$

$$(w,\eta) \longmapsto ([1,w], (\eta, \eta w))$$

well-defined

ii) Endow the Hermitian metric induced from $\mathbb{C}^2$

$$\|\xi\|^2 = |\xi z|^2 + |\xi|^2 = (1+|z|^2)|\xi|^2 ,$$

$$\|\eta\|^2 = (1+|w|^2)|\eta|^2 = \left(1 + \frac{1}{|z|^2}\right)|z|^2|\xi|^2$$

Use proj$\circ d$ from $\mathbb{C}^2$ as ∇

$$\left\langle (d(\xi z), d\xi), \frac{(z,1)}{\sqrt{1+|z|^2}} \right\rangle \frac{\boxed{(z,1)}}{\sqrt{1+|z|^2}}$$ ← coefficient of this

$$\Rightarrow \nabla\xi = \frac{1}{1+|z|^2}\Big(((d\xi)z + \xi dz)\bar{z} + d\xi\Big)$$

$$= d\xi + \underbrace{\boxed{\frac{\bar{z}\,dz}{1+|z|^2}}}_{=A_u}\xi$$

Since we do not use unitary trivialization, A is not skew-Hermitian (in $n=1$, purely-imaginary)

$$\mathring{A}_v = h^{-1} dh + h^{-1} \mathring{A}_u h \qquad h = g_{uv} \text{ for } L_{-1} = \omega$$

$$= \frac{dw}{w} + \frac{\bar{w}^{-1} d w^{-1}}{1 + \frac{1}{|w|^2}} = \frac{\bar{w}\, dw}{1+|w|^2}$$

iii) For L_n, $g_{uv} = h^{-n}$

Take $A_u = -n \mathring{A}_u$, $A_v = -n \mathring{A}_v$

would be a connection $\boxed{\text{check}}$

$$F_{A_u} = dA_u + \cancel{A_u \wedge A_u} \rightarrow 0 = -n\, d\left(\frac{\bar{z}\, dz}{1+|z|^2}\right)$$

$$= \frac{n\, dz \wedge d\bar{z}}{(1+|z|^2)^2}$$

$= 2dx\wedge dy = 2r\,dr\wedge d\theta$

$$\text{Consider } \int_{S^2} c(L_n) = n \int_{\mathbb{R}^2} \frac{i}{2\pi} \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} = n \int_0^\infty \frac{2r\, dr}{(1+r^2)^2}$$

$$= n \left. \frac{1}{1+r^2} \right|_{r=0}^{\infty} = n$$

iv) For $n \neq 0$, L_n and L_{-n} are not isomorphic

$ L_{-n} = L_n^*$ (transition = inverse + transpose)

2° i) $\mathbb{C}^n \longrightarrow E \xrightarrow{\pi} M$ with connection ∇

We can for the "conjugate" of E, $\bar{E}$ with $\bar{\nabla}$

$g^{\bar{E}}_{\alpha\beta} = \overline{g^{E}_{\alpha\beta}}$ and $A^{\bar{E}}_{\alpha} = \overline{A^{E}_{\alpha}}$

$\Rightarrow F^{\bar{\nabla}} = \overline{F^{\nabla}}$, $\det(\mathbb{1} + \frac{i}{2\pi} F^{\nabla}) = 1 + \sum_j c_j(E)$

take conjugate ↲ $\det(\mathbb{1} - \frac{i}{2\pi} \overline{F^{\nabla}}) = 1 + \sum_j c_j(E)$

$= 1 + \sum_{j\geq 0} (-1)^j c_j(\bar{E})$

Hence, $c_j(E) = (-1)^j c_j(\bar{E})$

ii) Endow E a bundle metric ⇝ transition is unitary $g^* = g^{-1}$

E^* has transition $(g^T)^{-1} = \bar{g} \Rightarrow E^* = \bar{E}$

§VI. Chern class and Pontryagin class

1° $\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^n$

For $\mathbb{R}^n \to E \xrightarrow{\pi} M$, we can form $E_{\mathbb{C}} = E \otimes \mathbb{C}$ which is a $\mathbb{C}^n$-bundle over M

A connection ∇ on E naturally extends to a connection on $E_{\mathbb{C}}$ by $\mathbb{C}$-linearity

Locally, $\nabla^E = d + A$. $\nabla^{E_{\mathbb{C}}} = d + A$

The only difference is the input: $\mathbb{R}^n$ or $\mathbb{C}^n$-valued

$\Rightarrow F_{\mathbb{C}} = F$

$$\det\left(\mathbb{1} + \frac{i}{2\pi} F_{\mathbb{C}}\right) = 1 + \sum_{j \geq 1} c_j(E_{\mathbb{C}})$$
$$= \det\left(\mathbb{1} + \frac{i}{2\pi} F\right) = 1 + \sum_{j \geq 1} (-1)^j p_j(E)$$
$$\Rightarrow \begin{cases} c_{2j+1}(E_{\mathbb{C}}) = 0 \\ c_{2j}(E_{\mathbb{C}}) = (-1)^j p_j(E) \end{cases}$$

2° i) On the other hand, given $\mathbb{C}^n \to E' \xrightarrow{\pi} M$

We can form $\mathbb{R}^{2n} \to E'_{\mathbb{R}} \xrightarrow{\pi} M$ by forgetting the complex structure

In terms of transition, $E'|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^n$, $g'_{\alpha\beta} \in Gl(n;\mathbb{C})$
$= \underset{\uparrow \atop \text{real}}{h_{\alpha\beta}} + \underbrace{i\, \underset{\uparrow \atop \text{imaginary}}{k_{\alpha\beta}}}$

$$g'_{\alpha\beta}(e_j) = h_{\alpha\beta} e_j + i k_{\alpha\beta} e_j$$

The basis for $\mathbb{R}^{2n}$ is taken to be $\{e_1, e_2, \dots, e_n, ie_1, \dots, ie_n\}$

$$\Rightarrow g_{\alpha\beta} = \begin{bmatrix} h_{\alpha\beta} & -k_{\alpha\beta} \\ k_{\alpha\beta} & h_{\alpha\beta} \end{bmatrix} \rightsquigarrow \text{define } E'_{\mathbb{R}}$$

$\boxed{\text{check}}$ $g_{\alpha\beta}$ is still invertible, and obeys the cocycle condition.

ii) $E'_{\mathbb{R}} \otimes \mathbb{C}$ is a $\mathbb{C}^{2n}$ vector bundle over M.

Transition is given by $g_{\alpha\beta} \curvearrowright \mathbb{C}^{2n} \ni \{v_1, \dots, v_{2n}\}$

Consider the change of basis $\{v_j - i v_{j+n}\}_{j=1}^{n} \cup \{v_j + i v_{j+n}\}_{j=1}^{n}$

$$g(v_j - i\, v_{j+n}) = (h_j^l v_l + k_j^l v_{n+l}) - i(-k_j^l v_l + h_j^l v_{n+l})$$

$$= (h_j^l + i k_j^l)(v_l - i\, v_{n+l}) = g'(v_l - i\, v_{n+l})$$

Similarly, $g(v_j + i\, v_{j+n}) = \bar{g}'(v_l + i\, v_{n+l})$

Hence, in this basis, transition for $E'_{\mathbb{R}} \otimes \mathbb{C}$ is $\begin{bmatrix} g'_{\alpha\beta} & 0 \\ 0 & \bar{g}'_{\alpha\beta} \end{bmatrix}$

$$\Rightarrow E'_{\mathbb{R}} \otimes \mathbb{C} = E' \oplus \bar{E}'$$

iii) $c(E'_{\mathbb{R}} \otimes \mathbb{C}) = 1 + \sum_{j \geq 1} c_j(E'_{\mathbb{R}} \otimes \mathbb{C})$

$$\|$$

$$c(E' \oplus \bar{E}') = c(E') \wedge c(\bar{E}')$$ homework

$$= (1 + c_1(E') + c_2(E') + \cdots) \wedge (1 - c_1(E') + c_2(E') \mp \cdots)$$

$$= 1 + (2c_2(E') - (c_1(E'))^2) + \cdots$$ (see previous section)

$$= 1 - p_1(E'_{\mathbb{R}}) + p_2(E'_{\mathbb{R}}) \mp \cdots$$

$$\Rightarrow p_1(E'_{\mathbb{R}}) = (c_1(E'))^2 - 2c_2(E')$$

for a complex vector bundle E'