§I. derivative

$E \xrightarrow{\pi} M$ vector bundle. $\Gamma(E) = \{$ smooth sections $\}$

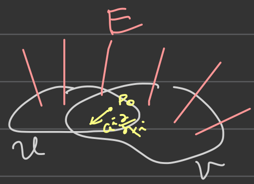
$\sim$ locally defined $\mathbb{R}^n$-valued functions

question taking derivative on $s \in \Gamma(E)$?

1° $E|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R}^n$ $\quad$ s is locally given by $\xi : \mathcal{U} \to \mathbb{R}^n$

$(p, \xi(p))$ $\quad$ For a tangent vector field $v^i \frac{\partial}{\partial x^i}$.

what about $v^i \frac{\partial}{\partial x^i} \xi$?



For another trivialization. $E|_{\mathcal{V}} \cong \mathcal{V} \times \mathbb{R}^n$

s is given by $\eta(p) = g_{vu}(p)\, \xi(p)$

$$\Rightarrow \underbrace{v^i \frac{\partial \eta}{\partial x^i}}_{} = g_{vu} \cdot \underbrace{v^i \frac{\partial \xi}{\partial x^i}}_{} + \underbrace{v^i \frac{\partial g_{vu}}{\partial x^i}}_{(*)} \xi$$

not a well-defined element of E_{P_0} $\Leftarrow$ (*) needs not be zero

another view point : derivative = limit of difference quotient

no canonical way to identify E_p and E_q for $p \neq q$

2° recall The Levi-Civita connection for TM

$$\nabla_{\frac{\partial}{\partial x^i}} \left(a^j \frac{\partial}{\partial x^j} \right) = \left(\frac{\partial a^j}{\partial x^i} + \Gamma^j_{ik} a^k \right) \frac{\partial}{\partial x^j}$$

besides derivative, there is a 0^{th}-order part

Formally, $\begin{cases} \nabla_{fX} Y = f\, \nabla_X Y \\ \nabla_X (fY) = f\, \nabla_X Y + X(f)\, Y \end{cases}$

lesson : the trouble term (*) is re-assembled into the 0^{th}-order part (Γ^k_{ij}).

§II. connection

defn a connection is a $\mathbb{R}$-linear map $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$, where $\mathfrak{X}(M) = \Gamma(TM)$

with $\begin{cases} \nabla_{fX} s = f\, \nabla_X s \\ \nabla_X (fs) = f\, \nabla_X s + X(f)\, s \end{cases}$

(∇s) output: section of E

$\uparrow$ input: tangent vector

Or equivalently, $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$

with $\nabla(fs) = f\nabla s + df \otimes s$

(a section of $T^*M \otimes E$ already means that when plug into fX, f can be pushed out)

1° <u>existence</u> $\{\mathcal{U}_\alpha\}$: locally finite cover of M with

$$\varphi_\alpha : E|_{\mathcal{U}_\alpha} \xrightarrow{\simeq} \mathcal{U}_\alpha \times \mathbb{R}^n$$

(abuse the notation: apply $\mathcal{U}_\alpha \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ for sections)

Then $\nabla^\alpha = \varphi_\alpha^{-1} d(\varphi_\alpha \circ s)$ is a connection for $E|_{\mathcal{U}_\alpha}$

It is nothing more than the usual derivative by using the trivialization φ_α

Let $\{\rho_\alpha\}$ be the partition of unity subordinate to $\{\mathcal{U}_\alpha\}$

Define $\nabla s = \sum_\alpha \rho_\alpha \nabla^\alpha (s|_{\mathcal{U}_\alpha})$

$$\nabla(fs) = \sum_\alpha \rho_\alpha \nabla^\alpha (f u_\alpha) = \sum_\alpha \rho_\alpha \left(f \nabla^\alpha u + df \otimes u\right)$$

$$= f\nabla s + df \otimes \nabla s \qquad \text{since } \sum_\alpha \rho_\alpha \equiv 1$$

$\#$

<u>lemma</u> Any vector bundle admits a connection

2° ambiguity ? <u>prop</u> Suppose that ∇ and $\tilde{\nabla}$ are two connections on E, then $\tilde{\nabla} - \nabla \in \Gamma(T^*M \otimes \operatorname{End} E)$

In other words, the space of connections is an affine space modelled on $\Gamma(T^*M \otimes \operatorname{End} E)$, End E-valued 1-forms

<u>lemma</u> $\mathcal{U}$: some open set in the Euclidean space

$F : \mathcal{C}^\infty(\mathcal{U}; \mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathcal{U}; \mathbb{R}^m)$ satisfying

$F(fs) = f F(s) \quad \forall f \in \mathcal{C}^\infty(\mathcal{U}; \mathbb{R}^1)$ and $s \in \mathcal{C}^\infty(\mathcal{U}; \mathbb{R}^n)$

Then there exists a

$$T \in \mathcal{C}^\infty(\mathcal{U}; \underbrace{M(m \times n; \mathbb{R})}_{\cong\, \mathbb{R}^{mn}}) \quad \text{such that} \quad F(s) = T \cdot s$$

(F cannot involve differentiation or integration)

pf: Consider "constant functions" $e_1, \cdots, e_n \in \mathcal{C}^\infty(\mathcal{U}; \mathbb{R}^n)$
and $\tilde{e}_1, \cdots, \tilde{e}_m \in \mathcal{C}^\infty(\mathcal{U}; \mathbb{R}^m)$

Write $F(e_j) = \sum_\mu T_j^\mu(x)\, \tilde{e}_\mu$.

By the condition, $F(s)$ must be $\begin{bmatrix} T_1^1 & \cdots & T_n^1 \\ & \ddots & \\ T_1^m & \cdots & T_n^m \end{bmatrix} s$. ※

From the lemma to the proposition

$$(\tilde{\nabla} - \nabla)(fs) = (f\,\tilde{\nabla}s + df \otimes s) - (f\,\nabla s + df \otimes s)$$
$$= f(\tilde{\nabla} - \nabla s)$$

Hence, $\tilde{\nabla} - \nabla$ is not a differential operator ... ※

3. local computation

summary: Suppose that ∇ is a connection on E

Then, ∇ on $E|_\mathcal{U} \cong \mathcal{U} \times \mathbb{R}^n$ is equal to

$d + A_u$ where A_u is an $(n \times n)$-matrix of 1-forms on $\mathcal{U}$

(By the lemma, and d is a connection on $E|_u$)

transition ? $E|_\mathcal{V} \cong \mathcal{V} \times \mathbb{R}^n$, $\nabla = d + A_v$

$s \in \Gamma(E)$

$\rightsquigarrow s_u = g_{uv} s_v$ $\quad s_u \in \mathcal{C}^\infty(\mathcal{U}; \mathbb{R}^n)$, $s_v \in \mathcal{C}^\infty(\mathcal{V}; \mathbb{R}^n)$

over $\mathcal{U} \cap \mathcal{V}$

$\nabla s \in \Gamma(T^*M \otimes E)$

$$\rightsquigarrow (ds_u + A_u s_u) = g\,(ds_v + A_v s_v)$$
$$\Rightarrow d(g s_v) + A_u(g s_v) = g\, ds_v + g A_v s_v$$
$$= (dg) s_v + g\, ds_v + A_u g s_v$$

True for any s_v $\Rightarrow$ $dg + A_u g = g A_v$

$\Rightarrow$ $g^{-1} dg + g^{-1} A_u g = A_v$

§III. an example

$\mathbb{C} \to L \longrightarrow S^2 = \mathbb{C} \cup \{\infty\}$

1° $S^2 = \mathbb{C} \cup \mathbb{C} \,/\, z = w^{-1}$

consider $g_{uv} = w = z^{-1} \ : \mathcal{U} \cap \mathcal{V} \longrightarrow Gl(1;\mathbb{C}) = \mathbb{C} \setminus \{0\}$

2° Let $A_u = \dfrac{\bar z\, dz}{1+|z|^2}$, $A_v = \dfrac{\bar w\, dw}{1+|w|^2}$

They have to be well-defined over $\mathcal{U}$ and $\mathcal{V}$, respectively

3° $$g^{-1}dg + g^{-1}A_u\, g = w^{-1}dw + \cancel{w^{-1}}\frac{\bar z\, dz}{1+|z|^2}\cancel{w}$$
$$= \frac{dw}{w} + \frac{1}{\bar w}\frac{-dw}{w^2}\frac{|w|^2}{1+|w|^2}$$
$$= \bar w\, dw \frac{1}{|w|^2}\left(1 - \frac{1}{1+|w|^2}\right) = \frac{\bar w\, dw}{1+|w|^2}$$

<u>remark</u> Neither A_u nor A_v is globally defined.

§IV. digression : End E

1° E is given by $\{\mathcal{U}_\alpha\}_\alpha$: open cover of M and
$g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow Gl(n;\mathbb{R})$ with the cocycle condition

$\rightsquigarrow \operatorname{End} E = E \otimes E^*$ transition is given by
$g_{\alpha\beta} \otimes (g_{\alpha\beta}^{-1})^T$ left acts on $\mathbb{R}^{n^2}$

2° a more convenient way for the fibers : $\mathbb{R}^{n^2} \cong M(n\times n;\mathbb{R})$

$B \in \Gamma(\operatorname{End} E)$, $s \in \Gamma(E) \Rightarrow Bs \in \Gamma(E)$

Suppose that s is given by $s_\alpha \in C^\infty(\mathcal{U}_\alpha;\mathbb{R}^n)$
B is given by $B_\alpha \in C^\infty(\mathcal{U}_\alpha; M(n\times n;\mathbb{R}))$

$$\begin{cases} s_\alpha = g_{\alpha\beta}\, s_\beta \\ B_\alpha s_\alpha = g_{\alpha\beta}\, B_\beta s_\beta \end{cases} \Rightarrow B_\alpha = g_{\alpha\beta} B_\beta\, g_{\alpha\beta}^{-1}$$

Hence, transition is $g_{\alpha\beta}(-)g_{\alpha\beta}^{-1}$ by using $M(n\times n;\mathbb{R})$ for $\mathbb{R}^{n^2}$

§V. curvature

1° As the Riemann curvature tensor, commuting covariant derivatives gives the curvature.

<u>defn</u> For a connection ∇ on E, its curvature is the tri-linear map

$$F^\nabla: \Gamma(M)\times\Gamma(M)\times\Gamma(E)\longrightarrow\Gamma(E)$$
$$(X,\ Y,\ s)\longmapsto \nabla_X\nabla_Y s-\nabla_Y\nabla_X s-\nabla_{[X,Y]}s$$

2° <u>lemma</u> F_∇ is $C^\infty(M;\mathbb{R})$-linear

and skew-sym here

According to the lemma last time, $F^\nabla\in\Gamma(E\otimes E^*\otimes T^*M\otimes T^*M)$

$$\Rightarrow F^\nabla\in\Gamma(\Lambda^2T^*M\otimes \operatorname{End}E)$$

(End E-valued 2-form, locally $n\times n$-matrix of 2-forms)

pf: In X and Y, similar to the Riemann curvature.

In s.

$$\begin{aligned}
&F^\nabla(X,Y,fs)\\
&=\nabla_X\nabla_Y(fs)-\nabla_Y\nabla_X(fs)-\nabla_{[X,Y]}(fs)\\
&=\nabla_X(f\nabla_Ys+Y(f)s)-\nabla_Y(f\nabla_Xs+X(f)s)\\
&\qquad-f\nabla_{[X,Y]}s-[X,Y](f)\cdot s\\
&=f\nabla_X\nabla_Ys+\cancel{X(f)\nabla_Ys}+\cancel{Y(f)\nabla_Xs}+\cancel{XY(f)s}\\
&\quad-f\nabla_Y\nabla_Xs-\cancel{Y(f)\nabla_Xs}-\cancel{X(f)\nabla_Ys}-\cancel{YX(f)s}\\
&\quad-f\nabla_{[X,Y]}s-\cancel{[X,Y](f)\cdot s}\\
&=f\,F^\nabla(X,Y,s)
\end{aligned}$$

$\cancel{\times}$

3° In terms of the local trivialization $E|_U\cong U\times\mathbb{R}^n$

Let $(x^1,x^2,\cdots)$ be the coordinate on U.

$\Rightarrow\ \nabla$ (over U) $=d+A_idx^i\qquad A_i\in C^\infty(U;M(n\times n;\mathbb{R}))$

$$F_{ij}=F^\nabla(\partial_i,\partial_j,-)\in M(n\times n;\mathbb{R})$$

$[\partial_i, \partial_j] \equiv 0$

$$F^\nabla(\partial_i, \partial_j, s) = (\partial_i + A_i)(\partial_j + A_j)(s) - (\partial_j + A_j)(\partial_i + A_i)(s) + 0$$

$$= \partial_i\partial_j s + (\partial_i A_j)s + \cancel{A_j(\partial_i s)} + \cancel{A_i(\partial_j s)} + A_i \cdot A_j \cdot s$$
$$- \partial_j \partial_i s - (\partial_j A_i)s - \cancel{A_i(\partial_j s)} - \cancel{A_j(\partial_i s)} - A_j \cdot A_i \cdot s$$

$F_{ij} =$

$$= (\partial_i A_j - \partial_j A_i + [A_i, A_j]) \cdot s$$

of course, no derivative on s

Usually write it as $\quad F^\nabla = \sum_{i,j} \frac{1}{2} F_{kl}\, dx^k \wedge dx^l$

$$(\Rightarrow F^\nabla(\partial_i, \partial_j) = \tfrac{1}{2} F_{ij} - \tfrac{1}{2} F_{ji} = F_{ij})$$

use $\wedge$ in matrix multiplication

4° $\quad F^\nabla = dA + A \wedge A$

right hand side $= d(A_j\, dx^j) + A_i\, dx^i \wedge A_j\, dx^j$

$$= \frac{\partial A_j}{\partial x^i} dx^i \wedge dx^j + A_i A_j\, dx^i \wedge dx^j$$

$$= \frac{1}{2}\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} + A_i A_j - A_j A_i\right) dx^i \wedge dx^j$$

We can double check that it is End E-valued

$$\tilde{A} = g^{-1} dg + g^{-1} A g \qquad (g = g_{uv},\ \tilde{A} = A_v,\ A = A_u)$$

<u>claim</u> $\quad g \tilde{F} g^{-1} = F$

$$\tilde{F} = d\tilde{A} + \tilde{A} \wedge \tilde{A}$$

$$= d(g^{-1} dg + g^{-1} A g) + (g^{-1} dg + g^{-1} A g) \wedge (g^{-1} dg + g^{-1} A g)$$

$$= dg^{-1} \wedge dg + \cancel{dg^{-1} \wedge A g} + \underline{g^{-1}\, dA\, g} - \cancel{g^{-1} A \wedge dg}$$

$$+ g^{-1} dg \wedge g^{-1} dg + \cancel{g^{-1} dg \wedge g^{-1} A g} + \cancel{g^{-1} A \wedge dg} + \underline{g^{-1} A \wedge A g}$$

$\left(\text{Since } g g^{-1} = \mathbb{1},\ (dg) g^{-1} + g\, d(g^{-1}) = 0 \right.$
$\left. \qquad dg^{-1} = -g^{-1} \cdot (dg) \cdot g \right)$

$$= g^{-1} F g$$

(compare with the discussion in § IV)

5° What does $d^2=0$ tell us?

$$O = d^2A = d(dA) = d(F - A\wedge A)$$

replace dA by $F - A\wedge A$

$$= dF - dA\wedge A + A\wedge dA$$

$$= dF - (F - A\wedge A)\wedge A + A\wedge(F - A\wedge A)$$

$\Rightarrow$ $\underline{dF = F\wedge A - A\wedge F}$ this is called the <u>Bianchi identity</u>.

§VI. re-visiting the example TS^2

0° $(M,g) \rightsquigarrow \nabla^{LC}$ is unique because of the torsion free condition

Except TM, this condition makes no sense.

1° $\mathbb{R}^2 \xrightarrow{F} S^2$

$(x,y) \longmapsto \frac{1}{1+x^2+y^2}(2x, 2y, 1-x^2-y^2)$ stereographic projection

$$|F_x| = \frac{2}{1+x^2+y^2} = |F_y| \quad , \quad F_x \perp F_y$$

2° By calculating $(F_{ij})^T$.

$$\nabla\left(\frac{1+x^2+y^2}{2}F_x\right) = \frac{2}{1+x^2+y^2}(y\,dx - x\,dy) \otimes \frac{1+x^2+y^2}{2}F_y$$

$$\nabla\left(\frac{1+x^2+y^2}{2}F_y\right) = \frac{2}{1+x^2+y^2}(-y\,dx + x\,dy) \otimes \frac{1+x^2+y^2}{2}F_x$$

With respect to $\left\{\frac{1+x^2+y^2}{2}F_x, \frac{1+x^2+y^2}{2}F_y\right\}$.

$$\nabla = d + \underbrace{\frac{2}{1+x^2+y^2}(y\,dx - x\,dy)}_{a} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

3° $$F = dA + A\wedge A = da\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \cancelto{0}{a\wedge a} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{4\,dx\wedge dy}{(1+x^2+y^2)^2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

4° $$\int_{S^2} \frac{4\,dx\wedge dy}{(1+x^2+y^2)^2} = \int_{S^2} K\,d\mathrm{vol}_g = 2\pi\cdot(2 - 2\,\mathrm{genus}) = 4\pi$$