§ I. vector bundle

tangent vector field ~ locally vector valued functions

1° a vector bundle over M basically means that there is a vector space E_p $\forall p \in M$

<u>defn 1</u> an n-dimensional vector bundle E over M consists of (E, π, M) where E : smooth manifold and $\pi : E \to M$ smooth map, and they satisfy

(i) for any $p \in M$, $E_p = \pi^{-1}(p)$ has a structure of an n-dimensional vector space

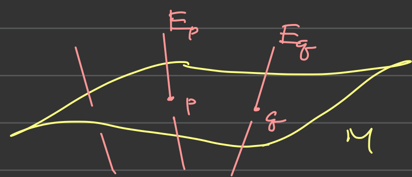
(ii) for any $p \in M$, there is an open neighborhood $\mathcal{U} \subset M$ such that $\pi^{-1}(\mathcal{U})$ is diffeomorphic to $\mathcal{U} \times \mathbb{R}^n$ (by $\varphi_{\mathcal{U}}$), and $\varphi_{\mathcal{U}}|_q : E_q \longrightarrow \mathbb{R}^n$ is a linear isomorphism.

locally trivial

<u>remark</u> we will also consider $\mathbb{C}^n$

<u>terminology</u>

$$\underset{\text{total space}}{E} \xrightarrow[\pi]{\text{projection}} \underset{\text{base}}{M}$$



2° an alternative way:

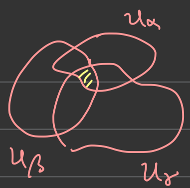
Suppose we have $\pi^{-1}(\mathcal{U}) \xrightarrow{\varphi_{\mathcal{U}}} \mathcal{U} \times \mathbb{R}^n$ and $\pi^{-1}(\mathcal{V}) \xrightarrow{\varphi_{\mathcal{V}}} \mathcal{V} \times \mathbb{R}^n$ with $\mathcal{U} \cap \mathcal{V} \neq \phi$.

$$\begin{array}{c} E \\ \cup \end{array}$$

$$\begin{tikzcd} (\mathcal{U}\cap\mathcal{V}) \times \mathbb{R}^n & \pi^{-1}(\mathcal{U}\cap\mathcal{V}) \arrow[l, "\varphi_{\mathcal{U}}"'] \arrow[r, "\varphi_{\mathcal{V}}"] & (\mathcal{U}\cap\mathcal{V}) \times \mathbb{R}^n \end{tikzcd}$$

$$(p, v) \longmapsto (p, g(p)\cdot v)$$

Since $\varphi_{\mathcal{V}} \circ \varphi_{\mathcal{U}}^{-1}$ is smooth and is a linear automorphism of $\{p\} \times \mathbb{R}^n$, $\varphi_{\mathcal{V}} \circ \varphi_{\mathcal{U}}^{-1}$ takes the form $(p, g(p)\cdot v)$ where $g : \mathcal{U} \cap \mathcal{V} \longrightarrow Gl(n; \mathbb{R})$



$p \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$

$$\begin{tikzcd}
& \mathbb{R}^n \arrow[ddr, bend left, "g_{\beta\gamma}"] & \\
& E_p \arrow[u, "\varphi_\alpha"'] \arrow[dl, "\varphi_\beta"'] \arrow[dr, "\varphi_\gamma"] & \\
\mathbb{R}^n \arrow[uur, bend left, "g_{\alpha\beta}"] & & \mathbb{R}^n \arrow[ll, bend left, "g_{\gamma\alpha}"]
\end{tikzcd}$$

$(p, g_{\alpha\beta}(p))$
$= \varphi_\alpha \circ \varphi_\beta^{-1}$

$\Rightarrow g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = \mathbb{1}$ $\quad$ ($\mathbb{1} \in Gl(n;\mathbb{R})$)

__defn 2__ a vector bundle over M consists of an open cover $\{\mathcal{U}_\alpha\}_\alpha$ together with smooth map $\{g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow Gl(n;\mathbb{R})\}$ such that $g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = \mathbb{1}$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$

the cocycle condition

(defn 1 $\Rightarrow$ defn 2 : as above. defn 2 $\Rightarrow$ defn 1 : quotient construction

$$E = \Big(\coprod_\alpha \mathcal{U}_\alpha \times \mathbb{R}^n \Big) \Big/ \underset{\in \mathcal{U}_\alpha \times \mathbb{R}^n}{(p, v)} \sim \underset{\in \mathcal{U}_\beta \times \mathbb{R}^n}{(p, g_{\alpha\beta}(p) v)} \qquad p \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$$

the cocycle condition $\Rightarrow E_p$ is indeed $\mathbb{R}^n$)

3° an analogous notion to tangent vector fields : we have a vector (in E_p) for each $p \in M$

__defn__ a section of $\pi : E \longrightarrow M$ is a smooth map $s : M \to E$ such that $\pi \circ s = \mathbb{1}_M$

Under
$$\begin{tikzcd}
& \pi^{-1}(\mathcal{U}) \arrow[r] & \mathcal{U} \times \mathbb{R}^n \\
s|_{\mathcal{U}} \arrow[ur] \arrow[urr] & & (p, s_u(p))
\end{tikzcd}$$
, a section is a collection of smooth, $\mathbb{R}^n$-valued functions $\{s_\alpha\}$ with $s_\alpha(p) = g_{\alpha\beta}(p) \cdot s_\beta(p)$ $\forall p \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$

Note that $s_\alpha \equiv 0$ $\forall \alpha$ is always defined, which is called the zero section.

§II. basic examples

1° M^n: manifold, TM: the tangent bundle

$(\mathcal{U}, (x^1, \cdots, x^n))$, $(\mathcal{V}, (y^1, \cdots, y^n))$ two coordinate chart

$$b^i \frac{\partial}{\partial y^i} = b^i \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j} = a^j \frac{\partial}{\partial x^j}$$

$$\Leftrightarrow \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix} = \begin{bmatrix} \frac{\partial x^1}{\partial y^1} & \cdots & \frac{\partial x^1}{\partial y^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x^n}{\partial y^1} & \cdots & \frac{\partial x^n}{\partial y^n} \end{bmatrix} \begin{bmatrix} b^1 \\ \vdots \\ b^n \end{bmatrix}$$

$$(k,l) = \frac{\partial x^k}{\partial y^l}$$

In terms of defn 2: $g^{TM}_{\alpha\beta} = \left[\frac{\partial x^i_\alpha}{\partial x^j_\beta} \right]$, cocyle: given by chain rule

Similarly, $\beta_i dy^i = \beta_i \frac{\partial y^i}{\partial x^j} dx^j = \alpha_j dx^j$

$$(k,l) = \frac{\partial y^l}{\partial x^k} \qquad \frac{\partial y^l}{\partial x^k} \frac{\partial x^k}{\partial y^m} = \delta^l_m \qquad \Rightarrow g^{T^*M}_{\alpha\beta}$$

(l,k) of inverse $\leftarrow$ (k,m)

= transpose inverse of $g^{TM}_{\alpha\beta}$

<u>remark</u> $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = \mathbb{1} \Rightarrow$ true for $((-)^T)^{-1}$

2° the tautological line bundle over $\mathbb{RP}^n$

$\mathbb{RP}^n = S^n / \pm 1$, parametrizes all the 1-dimensional vector subspaces in $\mathbb{R}^{n+1}$

$$L = \{ (p, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v \,/\!/\, p \}$$

When $n=1$, $\mathbb{RP}^1$ is still diffeomorphic to S^1

$\mathcal{U} = \{ [(x, \sqrt{1-x^2})] \mid x \in (-1, 1) \}$

$\mathcal{V} = \{ [(\sqrt{1-y^2}, y)] \mid y \in (-1, 1) \}$

coordinate transition? $\quad x>0, y>0 \Rightarrow y = \sqrt{1-x^2}$

$x<0, y<0 \Rightarrow y = -\sqrt{1-x^2}$

$$\left. \begin{array}{r} L|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R} \\ v = s(x, \sqrt{1-x^2}) \leftarrow (x, s) \\ L|_{\mathcal{V}} \cong \mathcal{V} \times \mathbb{R} \\ v = t(\sqrt{1-y^2}, y) \leftarrow (y, t) \end{array} \right\} \Rightarrow \frac{\sqrt{1-x^2}}{y} s = t \Rightarrow g_{\mathcal{V}\mathcal{U}} = \begin{cases} +1 & x \in (0,1) \\ -1 & x \in (-1, 0) \end{cases}$$

$\Rightarrow L$ is topologically the Möbius band

§III. constructions from the base

1° <u>restriction</u> $E \xrightarrow{\pi} M$. $Z \subset M$ submanifold

$\Rightarrow E|_Z = \pi^{-1}(Z) \subset Z$

In terms of defn'. $\tilde{U}_\alpha = U_\alpha \cap Z$. $\tilde{g}_{\alpha\beta} = g_{\alpha\beta}|_Z$

cocycle condition still holds

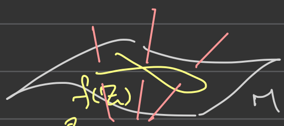
2° <u>pull-back</u>

$$\begin{array}{ccc} & & E \\ & & \downarrow \pi \\ Z & \xrightarrow{f} & M \end{array}$$

$f : Z \longrightarrow M$. smooth map.

$\Rightarrow f^*E$ is a vector bundle over Z

defined by $\{(p, e) \in Z \times E \mid f(p) = \pi(e)\}$



needs not to be a submanifold

In terms of defn'. $\begin{cases} \tilde{U}_\alpha = f^{-1}(U_\alpha) \\ \tilde{g}_{\alpha\beta} = g_{\alpha\beta} \circ f \end{cases}$

<u>remark</u> 1° is a special case of 2°. with the inclusion map

<u>example</u> For the constant map $c_{q_0} : \begin{array}{ccc} Z & \longrightarrow & M \\ p & \longmapsto & q_0 \end{array}$

$c_{q_0}^* E = Z \times \mathbb{R}^n$ is the trivial bundle

§IV. the quotient construction

1° <u>subbundle</u> $E \xrightarrow{\pi} M$, a submanifold $F \subset E$ is called a subbundle if $F_p = \pi^{-1}(p) \cap F$ is a vector subspace of E_p $\forall p \in M$

<u>discussion</u> (i) The definition implies that $\pi : F^{n+k} \to M^n$ is surjective and submersion

(ii) By the implicit function theorem, $F_p = E_p \cap F$ is a k-dimensional submanifold. Hence $F_p \cong \mathbb{R}^k$

(iii) $E|_U \overset{\varphi_U}{\cong} U \times \mathbb{R}^n$ Fix $p \in U$. we may assume $F_p \overset{\varphi_U}{\cong} \mathbb{R}^k \times \{0\}$

$F|_U \xrightarrow{\varphi_U} U \times \mathbb{R}^n \xrightarrow{pr} U \times \mathbb{R}^k$

Apply the implicit function theorem on $\varphi_U(F|_U) \xrightarrow{pr} U \times \mathbb{R}^k$ at $(p, 0)$ $\Rightarrow \varphi_U(F|_U) = (\underset{U}{q}, (\underset{\mathbb{R}^k}{v}, S(q, v)))$ near $(p, 0)$

$\in M(n-k \times k ; \mathbb{R})$

By the vector space structure $S(q, v) = \tilde{S}(q)\, v$

2° <u>quotient</u> bundle. Suppose that $F \subset E$ is a subbundle

We can form fiberwise quotient to get a vector bundle E/F

<u>discussion</u> (0) $E|_{\mathcal{U}} \overset{\varphi_{\mathcal{U}}}{\cong} \mathcal{U} \times \mathbb{R}^n \longleftarrow \mathcal{U}$ $\quad \{S_i\}_{i=1}^{n}$: (local) sections

$S_i \longleftarrow (p, e_i) \longleftarrow p$ and form a basis for E_p $\forall p \in \mathcal{U}$

They are called local trivializing sections.

These are equivalent to $\varphi_{\mathcal{U}}$

(i) As previous discussion $E|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R}^n$

construct trivializing sections for E/F

For a fixed $p \in \mathcal{U}$. $F_p \cong \{p\} \times (\mathbb{R}^k \times \{0\})$

Since F is a vector bundle, $F|_{\mathcal{U}}$ (shrink $\mathcal{U}$ if necessary)

admits local trivializing sections. $\{S_j\}_{j=1}^{k}$

such that $\varphi_{\mathcal{U}} \circ S_j(p) = (p, e_j)$

Since $\det \neq 0$ is an open condition

$\{\varphi_{\mathcal{U}} \circ S_1, \cdots, \varphi_{\mathcal{U}} \circ S_k, e_{k+1}, \cdots, e_n\}$ form a basis for E_q

for all q in some neighborhood of p

$\overset{\varphi_{\mathcal{U}}^{-1}}{\Rightarrow}$ $\{S_1, \cdots, S_k, S_{k+1}, \cdots, S_n\}$ local trivializing sections for E

and the first k-ones : for F

(ii) Suppose there is another such sections. $\{\tilde{S}_1, \cdots, \tilde{S}_n\}$

Since F is a subbundle

$$g^E_{vu} = \begin{bmatrix} g^E_{vu} & g''_{vu} \\ 0 & g'_{vu} \end{bmatrix}$$ takes this form

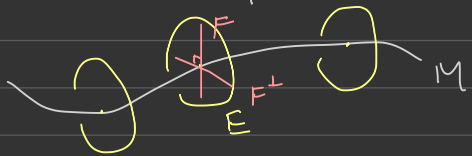
Hence, g'_{vu} must also satisfy the cocycle condition.

It gives the quotient bundle E/F

by partition of unity ...

3° With the help of metric. One can always choose a bundle metric on E. Namely, $\langle\ ,\ \rangle_p : E_p \times E_p \longrightarrow \mathbb{R}^1$ and depends on p smoothly

Then, $F^\perp$ is also a vector bundle, and is isomorphic to E/F



remark bundle isomorphism

$$\begin{array}{ccc} F_1 & \xrightarrow{\Phi} & F_2 \\ & \underset{\pi}{\searrow} \ \ \wr\!\wr \ \ \underset{\pi}{\swarrow} & \\ & M & \end{array}$$

$\Phi : F_1|_p \cong F_2|_p$

§V. algebraic constructions

(pseudo-) prop For vector bundles over M, one can perform linear algebraic operations on each E_p to get new vector bundles

pf: work out the corresponding g_{uv}

1° dual bundle

$$V_1 \xrightarrow{A} V_2$$

$$\mathrm{Hom}(V_1;\mathbb{R}) = V_1^* \xleftarrow{A^T} V_2^*$$

Over $\mathcal{U} \cap \mathcal{V}$

$$E|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R}^n \longrightarrow \mathcal{V} \times \mathbb{R}^n \cong E|_{\mathcal{V}}$$

$$(x, \xi) \longmapsto (x, g^E_{vu}(x)\xi)$$

$\rightsquigarrow$

$$E^*|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R}^n \longleftarrow \mathcal{V} \times \mathbb{R}^n \cong E^*|_{\mathcal{V}}$$

$$(x, (g^E_{vu}(x))^T\eta) \longmapsfrom (x, \eta)$$

$E^* = \mathrm{Hom}(E;\mathbb{R})$ is defined with the transition

$$g^{E^*}_{vu} = ((g^E_{vu})^T)^{-1}$$

2° direct sum $E_1 \oplus E_2$ $\quad g_{uv} = g^{E_1}_{uv} \oplus g^{E_2}_{uv}$, $\left[\begin{array}{c|c} g^{E_1} & 0 \\ \hline 0 & g^{E_2} \end{array}\right]$

tensor product. $E_1 \otimes E_2$ $\quad g_{uv} = g^{E_1}_{uv} \otimes g^{E_2}_{uv}$

$\Rightarrow \mathrm{Hom}(E, F) = F \otimes E^*$ ...