(the correct way to formulate "integration over manifold" and "fundamental theorem of calculus")

§I. exterior algebra

V: n-dimensional vector space over $\mathbb{R}$, V^*: dual space

$\Lambda^k V^* = \{$ alternating k-multi-linear maps $\}$

$\ni A : \underbrace{V \times \cdots \times V}_{k} \to \mathbb{R}^1$ linear on each factor

and $A(\cdots, v_i, \cdots, v_j \cdots)$

$= -A(\cdots, v_j, \cdots, v_i, \cdots)$

at any two factors

<u>lemma</u> $\dim \Lambda^k V^* = \begin{cases} \binom{n}{k} & \text{for } k \le n \\ 0 & \text{for } k > n \end{cases}$

Let $\{e_i\}_{i=1}^n$ be a basis for V and let $\{f^i\}_{i=1}^n$ be the dual basis. Namely, $f^j(e_i) = \delta_i^j$

$\rightsquigarrow f^{j_1} \wedge \cdots \wedge f^{j_k} \in \Lambda^k V^*$

As a map on $V \times \cdots \times V$, it sends $(v_1, \cdots, v_k)$ to

$$\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) f^{j_1}(v_{\sigma(1)}) f^{j_2}(v_{\sigma(2)}) \cdots f^{j_k}(v_{\sigma(k)})$$

S_k ↪ symmetric group

<u>example</u> $n=3$ $V^* = \operatorname{span}\{f^1, f^2, f^3\}$ dual to $\{e_1, e_2, e_3\}$

$\Lambda^1 V^* = V^*$

$\Lambda^2 V^* = \operatorname{span}\{f^1 \wedge f^2, f^2 \wedge f^3, f^3 \wedge f^1\}$

$\Lambda^3 V^* = \operatorname{span}\{f^1 \wedge f^2 \wedge f^3\} \cong \mathbb{R}^1$

For $v_1, v_2, v_3 \in V$, write $v_j = a_j^i e_i$

Then $f^1 \wedge f^2 \wedge f^3 (v_1, v_2, v_3) = \det(a_j^i)$

- $f^i \wedge f^i = 0$
- $f^{i_1} \wedge f^{i_2} = -f^{i_2} \wedge f^{i_1}$
- $\{f^{i_1} \wedge \cdots \wedge f^{i_k}\}_{i_1 < \cdots < i_k}$: basis for $\Lambda^k V^*$
- Formally, $\bigoplus_{k=0}^{n} \Lambda^k V^*$ is a graded algebra with "$\wedge$" being the multiplication

§II. exterior derivative

1° $\mathcal{U}$: open set of $\mathbb{R}^n$, $(x^1, \cdots, x^n)$ coordinate for $\mathbb{R}^n$

$\rightsquigarrow$ $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$: basis for $T_p\mathcal{U} \cong \mathbb{R}^n$ at any $p \in \mathcal{U}$

Denote the dual basis by $\{dx^j\}_{j=1}^n$

<u>defn</u> A (smooth) k-form on $\mathcal{U}$ is an assignment α

$p \in \mathcal{U} \longrightarrow \alpha_p \in \Lambda^k T_p^*\mathcal{U}$, and the assignment is smooth.

Explicity. $\alpha = \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$

$\hookrightarrow$ smooth functions on $\mathcal{U}$

2° exterior derivative

<u>defn</u> Define $d : \underbrace{\Omega^k(\mathcal{U})}_{=\{\text{smooth k-forms on } \mathcal{U}\}} \longrightarrow \Omega^{k+1}(\mathcal{U})$

$$\alpha = \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \longmapsto \sum_{j, i_1 < \cdots < i_k} \frac{\partial f_{i_1 \cdots i_k}}{\partial x^j}\, dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

<u>example</u> $\mathcal{U} = \mathbb{R}^3$

$\Omega^0 = \{f\}$

$\Omega^1 = \{p\,dx + q\,dy + r\,dz\}$

$\Omega^2 = \{a\,dy \wedge dz + b\,dz \wedge dx + c\,dx \wedge dy\}$

$\Omega^3 = \{g\,dx \wedge dy \wedge dz\}$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$ gradient

$$d(p\,dx + q\,dy + r\,dz) = P_x \cancel{dx \wedge dx}^{\,0} + P_y\, dy \wedge dx + P_z\, dz \wedge dx$$
$$+ q_x\, dx \wedge dy + q_z\, dz \wedge dy + r_x\, dx \wedge dz + r_y\, dy \wedge dz$$
$$= (r_y - q_z)\, dy \wedge dz + (P_z - r_x)\, dz \wedge dx$$
$$+ (q_x - P_y)\, dx \wedge dy$$ curl

$$d(a\,dy \wedge dz + b\,dz \wedge dx + c\,dx \wedge dy) = (a_x + b_y + c_z)\, dx \wedge dy \wedge dz$$ divergence

3° <u>lemma</u> i) $d^2 : \Omega^k \to \Omega^{k+2}$ is a zero map

ii) $\alpha \in \Omega^k$, $\beta \in \Omega^l$. $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

pf for i) Suppose $\alpha = f\, dx^1 \wedge \cdots \wedge dx^k$ ii) $\boxed{\text{exercise}}$

$$d\alpha = \frac{\partial f}{\partial x^i} dx^i \wedge dx^1 \wedge \cdots \wedge dx^k$$

$$d^2\alpha = \underbrace{\frac{\partial^2 f}{\partial x^j \partial x^i}}_{\text{symmetric in } (i,j)} \underbrace{dx^j \wedge dx^i}_{= -dx^i \wedge dx^j} \wedge dx^1 \wedge \cdots \wedge dx^k = 0 \quad ※$$

§ III. on manifold

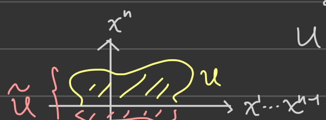
1° <u>recall</u> manifold : locally homeomorphic to open sets in $\mathbb{R}^n$ and transitions are diffeomorphism (smooth)

<u>defn</u> M is said to be a "manifold with boundary" if it is local homeomorphic to open sets in $\mathbb{H}^n = \{(x^1, \cdots, x^n) \mid x^n \geq 0\}$ and transitions are smooth*

interior points of M = $\{p \in M \mid \exists$ open neighborhood homeomorphic to open sets in $\mathbb{R}^n\}$

boundary points, $\partial M = M \setminus \{\text{interior points}\}$

<u>remark</u> * a function / map F defined on $\mathcal{U} \underset{\text{open}}{\subset} \mathbb{H}^n$ is smooth if $\exists\, \tilde{\mathcal{U}} \underset{\text{open}}{\subseteq} \mathbb{R}^n$, $\tilde{F}$: smooth on $\mathcal{U}$ such that $\mathcal{U} \subset \tilde{\mathcal{U}}$ and $\tilde{F}|_{\mathcal{U}} = F$



<u>lemma</u> ∂M is an $(n-1)$-dimensional manifold without boundary

(pf : DIY)

2° <u>defn</u> Similarly, a (smooth) k-form on M is an assignment α

$p \in \mathcal{U} \longrightarrow \alpha_p \in \Lambda^k T_p^* \mathcal{U}$, and the assignment is smooth.

In terms of coordinate cover, it means that

$$\begin{cases} \text{one has } f_{i_1 \cdots i_k}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \text{ on each chart} \\ f_{i_1 \cdots i_k}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \text{ and } h_{j_1 \cdots j_k}(y)\, dy^{j_1} \wedge \cdots \wedge dy^{j_k} \text{ coincides on } \mathcal{U} \cap \mathcal{V} \end{cases}$$

if and only if $f_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$

$$= h_{j_1 \cdots j_k}(y(x)) \frac{\partial y^{j_1}}{\partial x^{l_1}} \frac{\partial y^{j_2}}{\partial x^{l_2}} \cdots \frac{\partial y^{j_k}}{\partial x^{l_k}} dx^{l_1} \wedge \cdots \wedge dx^{l_k}$$

3° <u>defn</u> a manifold is said to be orientable if it admits a coordinate cover such that the Jacobian of the transition is positive definite everywhere (it is defined)

<u>remark</u> For regular surface in $\mathbb{R}^3$, it is equivalent to the previous definition. $\exists\ \nu$: smooth unit normal

(For each chart, by switching x^1, x^2, we may assume $\langle \frac{\partial X}{\partial x^1} \times \frac{\partial X}{\partial x^2}, \nu \rangle > 0$)

The definition has the advantage that it relies on the abstract structure of the manifold, but NOT $\Sigma \hookrightarrow \mathbb{R}^3$.

<u>lemma</u> M^n is orientable if and only if it admits a nowhere vanishing n-form.

sketch of proof : $\Rightarrow$) $\{U_i\}_{i=1}^{\infty}$ coordinate cover with det(Jacobian) > 0

$\hookrightarrow$ coordinate $(x_i^1, \cdots x_i^n)$

Let $\{\rho_i\}_{i=1}^{\infty}$: partition of unity subordinate to $\{U_i\}_{i=1}^{\infty}$

$\Rightarrow \sum_{i=1}^{\infty} \rho_i \, dx_i^1 \wedge \cdots \wedge dx_i^n$ would do the job

$\Leftarrow$) similar to the above remark ※

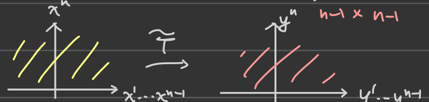
4° <u>lemma</u> If M : orientable, then ∂M is also orientable

<u>key lemma</u> $\tilde{T} : \mathbb{H}^n \circlearrowleft$ diffeomorphism and $\det(\text{Jac}(\tilde{T})) > 0$. (cannot be zero on $\partial\mathbb{H}^n$)

Then, $T = \tilde{T}|_{\mathbb{R}^{n-1}\times\{0\}}$ is also a diffeomorphism. and ($n \times n$)

$\det(\text{Jac}(T)) > 0$ ($n-1 \times n-1$)

pf :



$\tilde{T}$ extends to some open set ($y = y(x)$)

$y^n(x^1, \cdots, x^{n-1}, 0) = 0$ $\quad$ $\partial\mathbb{H}^n$ to $\partial\mathbb{H}^n$

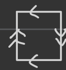
Look at $\det(\text{Jac}(\tilde{T})) > 0$ on $(x^1, \cdots, x^{n-1}, 0)$

$$= \det \begin{bmatrix} \text{Jac}(T) & * \\ 0 \cdots 0 & \frac{\partial y^n}{\partial x^n} \end{bmatrix} = \det(\text{Jac}(T)) \cdot \frac{\partial y^n}{\partial x^n} > 0$$

$\left.\frac{\partial y^n}{\partial x^1}\right|_{(x^1, \cdots, x^{n-1}, 0)} = 0 \longrightarrow 0 \cdots 0$

Since $\left.\frac{\partial y^n}{\partial x^n}\right|_{\partial \mathbb{H}^n} \geq 0$, $\det(Jac(T)) > 0$ and $\left.\frac{\partial y^n}{\partial x^n}\right|_{\partial \mathbb{H}^n} > 0$ ※

<u>example</u> i) Möbius band  is not orientable

ii) Klein bottle  is not orientable.

5° The following operators are well defined:

(i) $d : \Omega^k(M) \to \Omega^{k+1}(M)$

(ii) For $F : N \to M$, one has the pull-back map

$$F^* : \Omega^k(M) \to \Omega^k(M)$$, which commutes with d (d_M and d_N)

(iii) For a smooth vector field $V \in \mathfrak{X}(M)$, the contraction map

$$\iota_V : \Omega^k(M) \to \Omega^{k-1}(M)$$
$$\alpha \longmapsto \iota_V(\alpha)(W_1, \dots, W_{k-1}) = \alpha(V, W_1, \dots, W_{k-1})$$

<u>discussion</u> for (i) and (ii) $y = y(x)$ (either coordinate transition or a smooth map)

$$\begin{array}{ccc} f(y(x)) \frac{\partial y^1}{\partial x^{i_1}} \cdots \frac{\partial y^k}{\partial x^{i_k}} dx^{i_1} \wedge \cdots \wedge dx^{i_k} & \longleftarrow & f(y)\, dy^1 \wedge \cdots \wedge dy^k \\ \downarrow d & & \downarrow d \\ \left(\underbrace{\frac{\partial f}{\partial y^j} \frac{\partial y^j}{\partial x^l}}_{= \frac{\partial f(y(x))}{\partial x^l}} dx^l \right) \wedge \frac{\partial y^1}{\partial x^{i_1}} \cdots \frac{\partial y^k}{\partial x^{i_k}} dx^{i_1} \wedge \cdots \wedge dx^{i_k} & \longleftarrow & \frac{\partial f}{\partial y^j} dy^j \wedge dy^1 \wedge \cdots \wedge dy^k \end{array}$$

for (iii) If $\alpha = dx^1 \wedge \cdots \wedge dx^k$, $V = v^i \frac{\partial}{\partial x^i}$, $W_\mu = W_\mu^i \frac{\partial}{\partial x^i}$

$$\Rightarrow \alpha(V, W_1, \dots, W_{k-1}) = \det \begin{vmatrix} V^1 & W_1^1 & \cdots & W_{k-1}^1 \\ \vdots & \vdots & & \vdots \\ V^k & W_1^k & \cdots & W_{k-1}^k \end{vmatrix}$$

$$\Rightarrow \iota_V \alpha = V^1 dx^2 \wedge \cdots \wedge dx^k - V^2 dx^1 \wedge \widehat{dx^2} \wedge \cdots \wedge dx^k \pm \cdots + (-1)^{k-1} V^k dx^1 \wedge \cdots \wedge \widehat{dx^k}$$

§IV. integration and Stokes theorem

1° Since $\Lambda^n \mathbb{R}^n \cong \mathbb{R}^1$, given any two nowhere zero n-forms on M^n, α and $\tilde{\alpha}$, their quotient is a nowhere zero function

On an <u>oriented</u> manifold, define the integral of a (compactly supported) n-form, $\alpha = f(x)\, dx^1 \wedge \cdots \wedge dx^n$ to be the Lebesgue integral of $\int f\, dx^1 \cdots dx^n$ where $dx^1 \wedge \cdots \wedge dx^n$ is the positive

<u>discussion</u> $$h(y)\, dy^1 \wedge \cdots \wedge dy^n = \underbrace{h(y(x)) \det\left(\frac{\partial(y^1 \cdots y^n)}{\partial(x^1 \cdots x^n)}\right)}_{f(x)} dx^1 \wedge \cdots \wedge dx^n$$

$$\int h\, dy^1 \wedge \cdots \wedge dy^n = \int h(y)\, dy^1 \cdots dy^n$$

Lebesgue integrals

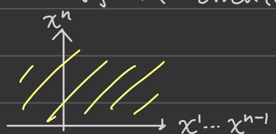
$$= \int h(y(x)) \left|\det\left(\frac{\partial(y^1 \cdots y^n)}{\partial(x^1 \cdots x^n)}\right)\right| dx^1 \cdots dx^n$$ change of variable formula

$$= \int \mathrm{sgn}\left(\det\left(\frac{\partial(y^1 \cdots y^n)}{\partial(x^1 \cdots x^n)}\right)\right) f(x)\, dx^1 \wedge \cdots \wedge dx^n$$

$\Rightarrow$ well-defined if $(x^1, \dots, x^n)$ and $(y^1, \dots, y^n)$ give the same orientation

2° (revisiting lemma of §III-4°): M = manifold with boundary

If M oriented, then ∂M is also oriented



By $x^1 \leftrightarrow -x^2$, we may assume that $dx^1 \wedge \cdots \wedge dx^n > 0$

<u>Defn</u> The induced orientation is defined to be $\iota\left(-\frac{\partial}{\partial x^n}\right) dx^1 \wedge \cdots \wedge dx^n$

$$= (-1)^n dx^1 \wedge \cdots \wedge dx^{n-1}$$

(well-defined: lemma §III-4°)

3° <u>Theorem</u> (Stokes) M^n: oriented manifold with boundary

∂M: with the induced orientation. Then

$$\int_{\partial M} \alpha = \int_M d\alpha$$ for any compactly supported $(n-1)$-form α <u>on M</u>

Note that for the L.H.S. it is $\alpha|_{\partial M}$, or pull-back of α by the inclusion $\partial M \hookrightarrow M$

pf of the <u>key step</u>: Suppose $\mathrm{supp}(\alpha) \subset$ one chart

$$\alpha = \sum_{j=1}^n (-1)^{j-1} f_j(x)\, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n \qquad f_j : \text{compact support}$$

$$\begin{aligned} \partial M &\hookrightarrow M \\ (x^1, \dots, x^{n-1}) &\mapsto (x^1, \dots, x^{n-1}, 0) \end{aligned} \Rightarrow dx^n|_{\partial M} = 0 \Rightarrow \alpha|_{\partial M} = (-1)^{n-1} f_n\, dx^1 \wedge \cdots \wedge dx^n$$

$$\alpha|_{\partial M} = (-1)^{n-1} f_n\, dx^1\wedge\cdots\wedge dx^n = -f_n(x^1,\dots,x^{n-1},0)\,\overbrace{\left((-1)^n dx^1\wedge\cdots\wedge dx^n\right)}^{\text{induced orientation}}$$

$$\Rightarrow \int_{\partial M}\alpha = -\int_{\mathbb{R}^{n-1}} f_n(x^1,\dots,x^{n-1},0)\, dx^1\cdots dx^{n-1}$$

$$d\alpha = \Big(\sum_j \frac{\partial f_j}{\partial x^j}\Big)\underbrace{dx^1\wedge\cdots\wedge dx^n}_{\text{correct orientation}}$$

$$\Rightarrow \int_M d\alpha = \int_{\mathbb{R}^n}\Big(\sum_{j=1}^{n-1}\frac{\partial f_j}{\partial x^j}\Big)dx^1\cdots dx^n + \Big(\frac{\partial f_n}{\partial x^n}\Big)dx^n\, dx^1\cdots dx^{n-1}$$

$$= 0 \;\boxed{\text{think}} \;+ \int_{\mathbb{R}^{n-1}}\Big(\cancelto{0}{f_n(x^1,\dots,x^{n-1},\infty)} - f_n(x^1,\dots,x^{n-1},0)\Big)\, dx^1\cdots dx^{n-1}$$

same

※

§V. de Rham cohomology

$\Omega^k(M)$ ($\sim$ smooth functions) is an ∞-dimensional vector space.

How to get some finite dimensional data?

See the difference between  and  ?

1° defn The k-th de Rham cohomology is defined by

$$H^k_{dR}(M) = \ker(d \text{ on } \Omega^k(M)) \big/ d\Omega^{k-1}(M)$$

subspace, since $d^2=0$

(dual concept intuitively: compact, boundaryless submanifold in M, which is not boundary of others)

2° Note that for a smooth map $F: M\to N$,

it follows from $F^*\circ d_N = d_M\circ F^*$ that

F^* induces a map from $H^k_{dR}(N)$ to $H^k_{dR}(M)$

It turns out that "similar" maps descend the same map on de Rham cohomologies.

<u>defn</u> $F_0, F_1 : M \to N$ are said to be homotopic if $\exists\ H : M \times [0,1] \to N$ such that $\begin{cases} H(-,0) = F_0 \\ H(-,1) = F_1 \end{cases}$

<u>prop</u> For homotopic maps, F_0 and F_1,

$$F_0^* = F_1^* \text{ on } H^k_{dR}(N) \text{ to } H^k_{dR}(M)$$

pf:

$$M \underset{\iota_1}{\overset{\iota_0}{\rightrightarrows}} \underset{t}{M \times [0,1]} \xrightarrow{H} N$$

$$\iota_j(p) = (p, j) \Rightarrow H \circ \iota_j = F_j, \quad j = 0, 1$$

t-dependent k-form on M ... $(k-1)$ ...

$$\xi \in \Omega^k(N), \quad H^*(\xi) = \alpha(t) + dt \wedge \beta(t)$$

$\alpha(t) = a_{i_1 \cdots i_k}(x, t)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$

$$\Lambda^k T^*(M \times [0,1]) = \Lambda^k T^*M \oplus \underbrace{\Lambda^{k-1} T^*M}_{\wedge dt}$$

Note that $F_0^*(\xi) = \alpha(0)$, $F_1^*(\xi) = \alpha(1)$

If $d\xi = 0$, $0 = H^*(d\xi) = d(H^*(\xi))$

$$= d_M \alpha(t) + dt \wedge \frac{\partial \alpha}{\partial t} - dt \wedge d_M \beta(t)$$

$$\Rightarrow d_M \alpha(t) = 0 \text{ and } \frac{\partial \alpha}{\partial t} = d_M \beta(t)$$

Hence, $F_1^*(\xi) - F_0^*(\xi) = \int_0^1 \frac{\partial \alpha}{\partial t}\, dt = \int_0^1 (d_M \beta(t))\, dt$

$$= d_M \left(\int_0^1 \beta(t)\, dt \right)$$

trivial in $H^k_{dR}(M)$ //

3° <u>lemma</u> (Poincaré) $\begin{cases} H^0_{dR}(\mathbb{R}^n) = \mathbb{R}^1 = \{\text{constant functions}\} \\ H^{j \neq 0}_{dR}(\mathbb{R}^n) = 0 \end{cases}$

pf: $n = 1$ $\quad \Omega^1(\mathbb{R}^1) = \ker(d \text{ on } \Omega^1(\mathbb{R}^1)) = \{ f(x)\, dx \} = d\Omega^0$

Let $F(x) = \int_0^x f(\xi)\, d\xi \Rightarrow dF = f(x)\, dx$

$n > 1$

$$\underset{(x^1, \cdots, x^{n-1}, x^n)}{\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}^1} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \underset{(x^1, \cdots, x^{n-1})}{\mathbb{R}^{n-1}}$$

ι : to $\mathbb{R}^{n-1} \times \{0\}$

$$\Rightarrow \pi \circ \iota = \mathbb{1}_{\mathbb{R}^{n-1}} \Rightarrow \iota^* \circ \pi^* = \mathbb{1} \text{ on } H^k_{dR}(\mathbb{R}^{n-1})$$

$\iota \circ \pi(x^1, \ldots, x^{n-1}, x^n) = (x^1, \ldots, x^{n-1}, 0) : \mathbb{R}^n \circlearrowleft$

Note that $\iota \circ \pi$ is homotopic to $\mathbb{1}_{\mathbb{R}^n}$ by

$$\begin{aligned} H : \mathbb{R}^n \times [0,1] &\longrightarrow \mathbb{R}^n \\ ((x^1, \ldots, x^{n-1}, x^n), t) &\longmapsto (x^1, \ldots, x^{n-1}, t x^n) \end{aligned} \Rightarrow \begin{cases} H(-,0) = \iota \circ \pi \\ H(-,1) = \mathbb{1}_{\mathbb{R}^n} \end{cases}$$

Hence, $\underbrace{(\iota \circ \pi)^*}_{\pi^* \circ \iota^*} = \mathbb{1}$ on $H^k_{dR}(\mathbb{R}^n)$

$$\Rightarrow \quad H^k_{dR}(\mathbb{R}^n) \underset{\iota^*}{\overset{\pi^*}{\rightleftarrows}} H^k_{dR}(\mathbb{R}^{n-1}) \quad \text{is an isomorphism} \quad \#$$