c.f. Lawson & Osserman 1977

§I. minimal graph in higher codimensions

1° $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m \rightsquigarrow \Gamma_F = \{ (x, F(x)) \in \mathbb{R}^{n+m} \}$

Γ_F is minimal if and only if

$$\frac{\partial}{\partial x^i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial F}{\partial x^j} \right) = 0 \quad \text{—— } (*) \qquad g_{ij} = \delta_{ij} + \langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \rangle$$

in $\mathbb{R}^m$

The system consists of m equations

2° known results for $m = 1$ (hypersurface case)

i) Γ_F is a volume minimizer

ii) The Dirichlet is solvable for arbitrary continuous data:
Given $\phi : \partial\Omega \to \mathbb{R}^m$, there exists $F : \Omega \to \mathbb{R}^m$ satisfying $(*)$ and $F|_{\partial\Omega} = \phi$. Moreover, F is unique and real-analytic

(by Jenkins & Serrin, de Giorgi, Morrey, etc.)

3° <u>theorem</u> (monotonicity formula)

$\Sigma^n \subset \mathbb{R}^N$, minimal

Then, for any $p \in \Sigma$, $\mathrm{Vol}(\Sigma \cap B(p;r)) \geq C_n r^n \quad \forall r \geq 0$

C_n : Volume of unit ball in $\mathbb{R}^n$

(pf: same as the $N = n+1$ case)

demonstrate non-minimal example in class

4° <u>lemma</u> $\tilde{\Sigma}^n \subset \mathbb{R}^{n+k}$, compact, minimal with non-empty boundary

Then, $\mathrm{Vol}(\tilde{\Sigma}) = \frac{1}{n} \int_{\partial\tilde{\Sigma}} \langle X, \eta \rangle$

η → unit out conormal of $\partial\tilde{\Sigma}$

X → position vector field

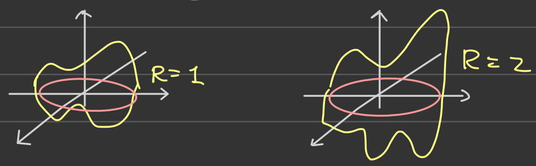
pf:
$$\begin{aligned}
\mathrm{div}(X^T) &= \sum_i \langle \nabla_{e_i} X^T, e_i \rangle \\
&= \sum_i \langle e_i(X^T), e_i \rangle \\
&= \sum_i \langle e_i(X), e_i \rangle - \langle e_i(X^\perp), e_i \rangle \\
&= \sum_i \langle e_i, e_i \rangle + \langle X^\perp, (e_i(e_i))^\perp \rangle = n + \langle X^\perp, \cancel{H} \rangle^{0}
\end{aligned}$$

apply divergence theorem ⊛

§ II. non-existence for the Dirichlet problem

<u>prop</u> $\phi : \partial B^n = S^{n-1} \longrightarrow S^{m-1} \subset \mathbb{R}^m$, which is NOT homotopic to a constant map. Suppose that $n > m$. Then, there exists a constant $R_\phi > 0$ such that the Dirichlet problem for $(*)$ with $\phi_R = R \cdot \phi$ admis no solution when $R \geq R_\phi$

<u>remark</u> The rescaling only acts on the $\mathbb{R}^m$ part



pf: Suppose the Dirichlet problem for $(*)_m$ admits a solution on B^n

i) $B^n \xrightarrow{F} \mathbb{R}^m$ Suppose that $F(x) \neq 0$ $\forall x \in B^n$

$$\begin{array}{ccc} & B^n \xrightarrow{F} \mathbb{R}^m & \\ \nearrow & & \searrow \\ [0,1] \times S^{n-1} & & S^{m-1} \\ (r, x) & \longmapsto & F(ry)/|F(ry)| \end{array}$$

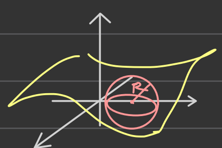
homotopic between ϕ and $F(0)/|F(0)|$ (as maps from S^{n-1} to S^{m-1})

It contradicts to that ϕ is not null homotopic.

Hence, $\exists\, x_0 \in B^n$ such that $F(x_0) = 0$

$\Rightarrow (x_0, 0) \in \Gamma_F$

ii) $\mathrm{dist}^2((x_0, 0), \partial\Gamma_F) = \min\limits_{|x|=1} (x_0 - x)^2 + R^2 \underbrace{\phi^2(x)}_{} \geq R^2$, $\phi^2(x) = 1$



By the theorem (monotonicity formula)

$\underline{\mathrm{Vol}(\Gamma_F)} \geq \mathrm{Vol}(\Gamma_F \cap B((x_0, 0); R)) \geq \underline{C_n R^n}$

iii) $\partial\Gamma_F = \{ (x, R\phi(x)) \mid |x| = 1 \}$

$\mathrm{Vol}(\partial\Gamma_F) = ?$ choose $s^1, \cdots, s^{n-1}$: local coordinate for S^{n-1}

$g_{ij}(\text{of } S^{n-1}) = \langle \frac{\partial x}{\partial s^i}, \frac{\partial x}{\partial s^j} \rangle \leftarrow \mathbb{R}^n$

$\tilde{g}_{ij}(\text{of } \partial\Gamma_F) = \underbrace{\langle \frac{\partial x}{\partial s^i}, \frac{\partial x}{\partial s^j} \rangle}_{\mathbb{R}^n} + R^2 \underbrace{\langle \frac{\partial \phi}{\partial s^i}, \frac{\partial \phi}{\partial s^j} \rangle}_{\mathbb{R}^m}$

$\Rightarrow \tilde{g}_{ij} = g_{ij} + R^2 \langle \frac{\partial \phi}{\partial s_i}, \frac{\partial \phi}{\partial s_j} \rangle \rightsquigarrow$ study $\det(\tilde{g}_{ij})$ pointwisely (at x_0)

Apply singular value decomposition to $D\phi$ @ x_0

$\exists \begin{cases} e_1, \cdots, e_{m-1}, \underbrace{e_m, \cdots, e_{n-1}}_{n-m} & \text{orthonormal basis for } T_{x_0} S^{n-1} \\ v_1, \cdots, v_{m-1}, & \text{orthonormal basis for } T_{\phi(x_0)} S^{m-1} \end{cases}$ use $n>m$ here

such that $D\phi(e_j) = \begin{cases} \lambda_j e_j & j = 1, \cdots, m-1 \\ 0 & j = m, \cdots, n-1 \end{cases}$

($\{e_j\}$: eigenbasis of $(D\phi)^T D\phi : T_{x_0} S^{n-1} \circlearrowleft$)

$\Rightarrow g_{ij}(x_0) = \delta_{ij}$, $\tilde{g}_{ij}(x_0) = (1 + \lambda_i^2 R^2)\, \delta_{ij}$ (set $\lambda_{i \geq m} = 0$)

Hence $\det(\tilde{g}_{ij}) \big/ \det(g_{ij}) = 1 + R^{2m-2} \prod_{i=1}^{m-1} |\lambda_i|$

$\Rightarrow \mathrm{Vol}(\partial \Gamma_F) \leq \tilde{c} R^{m-1}$ for $R \geq 1$

iv) By the lemma

$\underline{\mathrm{Vol}(\Gamma_F)} = \frac{1}{n} \int_{\partial \Gamma_F} \langle X, \eta \rangle \leq \frac{1}{n} \int_{\partial \Gamma_F} |X| \leq \underline{\frac{\tilde{c}}{n} R^{m-1} \sqrt{1+R^2}}$

on $\partial \Gamma_F$, $X = (x, R\phi(x)) \Rightarrow |X| = \sqrt{1+R^2}$

v) To sum up, $c_n R^n \leq \frac{\tilde{c}}{n} R^{m-1} \sqrt{1+R^2}$

Since $n > m$, this cannot be true for large R ※

§ III. the main example (Lawson — Osserman cone, $n=4$, $m=3$)

Consider the Hopf map, $\phi : S^3 \subset \mathbb{C}^2 \longrightarrow S^2 \subset \mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$

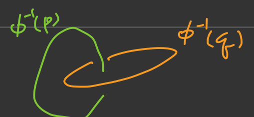
$(z, w) \longmapsto (|z|^2 - |w|^2, 2\bar{z}w)$

$(|z|^2 - |w|^2)^2 + 4|z|^2|w|^2 = (|z|^2 + |w|^2)^2 = 1$

1° Hopf map is NOT null homotopic

<u>key</u> choose any two $p \neq q \in S^2$.

Look at $\phi^{-1}(p)$ and $\phi^{-1}(q)$ in $S^3 = \mathbb{R}^3 \cup \{\infty\}$



They have "linking number" ± 1

If ϕ is null homotopic, linking number is zero.

$\phi^{-1}(p),\ \phi^{-1}(q)$: disjoint S^1 in $\mathbb{R}^3$ $\quad\Rightarrow\quad \frac{x-y}{|x-y|} : S^1\times S^1 \to S^2$

$\quad\ \ \overset{\Downarrow}{x} \qquad \overset{\Downarrow}{y}$ $\qquad$ linking number = degree of this map.

2° explicit calculation

$$\partial P_F = \{ (x, R\phi(x)) \mid x\in S^3 \}$$

at $N = (1,0,0,0)$ $\qquad \{e_j\}_{j=2}^{3} \in T_N S^3$

e_2 : $z = 1 + it + O(t^2)$, $w = O(t^2)$ $\qquad \partial_2(x, R\phi(x)) @ N = (e_2, (0,0,0))$

e_3 : $z = 1 + O(t^2)$, $w = t + O(t^2)$ $\qquad \partial_3(x, R\phi(x)) @ N = (e_3, (0, 2R, 0))$

e_4 : $z = 1 + O(t^2)$, $w = it + O(t^2)$ $\qquad \partial_4(x, R\phi(x)) @ N = (e_4, (0,0,2R))$

Due to "$SU(2)$ symmetry" of the Hopf map,

$\det(\tilde g_{ij}) / \det(g_{ij})$ is a constant $\leftarrow$ let us assume this

$= (1+4R^2)^2$

$$\Rightarrow\ \underbrace{\tfrac{1}{2}\pi^2}_{\mathrm{Vol}(B^4)} R^4 \le \mathrm{Vol}(P_F) \le \underbrace{2\pi^2}_{\mathrm{Vol}(S^3)}\,\frac{1}{4}\sqrt{1+R^2}\,(1+4R^2)$$

$\Rightarrow$ $R \le 4.1732$. If $R > 4.18$, the Dirichlet problem for $(*)$ with $R\phi$ has no solution

<u>remark</u> By perturbing the n-plane (inverse function theorem), one can show the existence of solution for $R \ll 1$

3° The minimal cone?

For the cone for $R\phi$, $\quad C_R = \{ (x, R|x|\phi(\frac{x}{|x|})) \mid x\in B^4 \}$

$\left(\begin{array}{l} \Sigma \subset S^N \qquad C(\Sigma) = \{ rp \in \mathbb{R}^{N+1} \mid r>0,\ p\in\Sigma \} \\ \boxed{\text{check}}\quad \Sigma : \text{minimal in } S^N \Leftrightarrow C(\Sigma) : \text{minimal in } \mathbb{R}^{N+1} \end{array} \right)$

Will C_R be minimal? $\quad C_R \cap S^6 \subset \mathbb{R}^7$

$$\overset{\shortparallel}{\Sigma_s = \{ (sx, \sqrt{1-s^2}\,\phi(x)) \mid |x| = 1 \}} \qquad s = \frac{\sqrt{1-R^2}}{R}$$

If Σ_s is minimal in S^6, at least the volume is critical among "s"

According to previous calculation @ N

$$\partial_2 \mapsto (s e_2 + (0,0,0))$$

$$\partial_3 \mapsto (s e_3 + 2\sqrt{1-s^2}\,(0,1,0)) \qquad |\text{length}|^2 = s^2 + 4(1-s^2) = 4-3s^2$$

$$\partial_4 \mapsto (s e_4 + 2\sqrt{1-s^2}\,(0,0,1))$$

$$\Rightarrow \mathrm{Vol}(\Sigma_s) = 2\pi^2 s(4-3s^2)$$

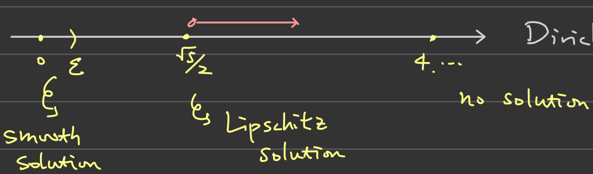
$$\frac{d}{ds}\mathrm{Vol}(\Sigma_s) = 2\pi^2(4-9s^2) \quad \Rightarrow \quad s = \pm 2/3$$

It turns out it gives a minimal S^3 in S^6.

$\boxed{\text{check}}$ $\left\{ \left(\frac{2}{3}x, \frac{\sqrt{5}}{3}\phi(x)\right) \,\middle|\, |x|=1 \right\}$ is minimal in S^6

Hence, its cone $C_{\underbrace{\frac{\sqrt{5}}{2}}_{=1.118\ldots}} = \left\{ \left(x, \frac{\sqrt{5}}{2}|x|\,\phi\!\left(\frac{x}{|x|}\right)\right) \,\middle|\, 0<|x|\le 1 \right\}$

4° summary and remark $\quad \phi: S^3 \to S^2$ Hopf map



Dirichlet problem with $R\phi$

Conjecture: When $R > \sqrt{5}/2$, no solution

remark: The cone is volume minimizer (Harvey & Lawson 1982)

When $R > \sqrt{5}/2$, $\exists$ minimal submanifold in $\mathbb{R}^7$ with boundary $= \{(x, R\phi(x)) \mid |x|=1\}$

But the topology is not a 4-ball.