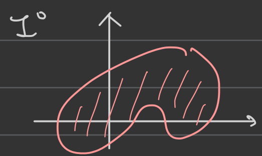
§I. motivation

1°

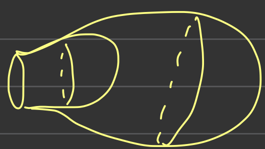


$\Omega \subset \mathbb{R}^2$. a region (topologically disk)

isoperimetric inequality: $4\pi \underbrace{A(\Omega)}_{\text{area}} \le (\underbrace{L(\partial\Omega)}_{\text{boundary}})^2$

"=" if and only if Ω is a disk

2° What about surface in $\mathbb{R}^3$? It cannot be true in general.



$\rightsquigarrow$ $A(\Omega)$ can be arbitrarily large.

We have to impose some condition on the surface.

<u>conjecture</u> $\Omega^n \subset \mathbb{R}^{n+m}$ compact, minimal, with non-empty boundary

$\Rightarrow$ $\mathrm{Vol}(\Omega)^{n-1} \le C_n \, \mathrm{Vol}(\partial\Omega)^n$

where $C_n = \dfrac{\mathrm{Vol}(B^n)^{n-1}}{\mathrm{Vol}(\partial B^n)^n}$

<u>result</u> 0) true in general, but with a larger C_n

<u>sharp C_n?</u> i) Almgren 1986: true for volume minimizer

ii) Carleman 1921: true when $n=2$ (and disk type)

iii) $k=0$: minimal condition = no condition

$\mathrm{Vol}(\Omega)$ = Lebesgue measure $\rightsquigarrow$ measure theoretical argument

(c.f. Osserman 1978. by Brunn–Minkowski theorem)

iv) Brendle 2018: true when $m \le 2$

3° minimal submanifold (in general)

$\Sigma \subset (M, g)$ $\qquad A(X,Y): T\Sigma \times T\Sigma \longrightarrow N\Sigma$

$$(X, Y) \longmapsto (\nabla^M_X Y)^\perp$$

$\vec{H} = \mathrm{tr}_g(A)$ is a normal vector field of Σ in M

It is called the mean curvature vector.

For any normal vector field V, $\left.\frac{d}{dt}\right|_{t=0} \mathrm{Vol}(\Sigma + tV) = -\int_\Sigma \langle V, \vec{H}\rangle$

<u>defn</u> $\vec{H} \equiv 0$ is called a minimal submanifold

(similar computation)

§II. surface case (c.f. Li & Schoen & Yau 1984)

change notation here

1° $\Sigma^2 \subset \mathbb{R}^m$ $\quad X(x^1, x^2) = (w^1(x^1,x^2), \cdots, w^m(x^1,x^2))$

$$\Rightarrow \frac{\partial^2 X}{\partial x^i \partial x^j} = \Gamma_{ij}^k \frac{\partial X}{\partial x^k} + A(\partial_i, \partial_j) \in T\Sigma \oplus N\Sigma$$

$$\Rightarrow \Delta X = g^{ij}\left(\frac{\partial^2 X}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial X}{\partial x^k}\right) = g^{ij} A(\partial_i, \partial_j) = \vec{H}$$

Still. Σ is minimal if and only if coordinate functions are harmonic functions

2° <u>lemma</u> (Wirtinger's inequality) $f(\theta) : S^1 \to \mathbb{R}^1$. smooth with $\int_0^{2\pi} f(\theta)\, d\theta = 0$ Then, $\int_0^{2\pi} f^2\, d\theta \leq \int_0^{2\pi} (f')^2\, d\theta$

pf: By Fourier $f(\theta) = \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} (a_n \sin(n\theta) + b_n \cos(n\theta))$

No constant term due to $\int_0^{2\pi} f\, d\theta = 0$

$$f'(\theta) = \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} n(a_n \cos(n\theta) - b_n \sin(n\theta))$$

$$\Rightarrow \int_0^{2\pi} f^2\, d\theta = \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2) = \int_0^{2\pi} (f')^2\, d\theta \quad \divideontimes$$

3° Now, assume Σ is topologically a disk.

$\Rightarrow \partial\Sigma$ is a circle, in particular, connected

By translation, we may assume $\int_{\partial\Sigma} w^i\, ds = 0 \quad \forall i = 1, \cdots, m$

arc-length parameter

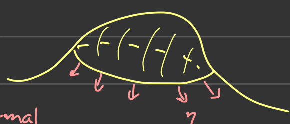
Consider $r^2 = \sum_{i=1}^{m} (w^i)^2$ = distance^2 to the origin

$$\Delta r^2 = \sum_{i=1}^{m} w^i \underbrace{\Delta w^i}_{\substack{= 0 \\ \text{by minimal}}} + 2\sum_{i=1}^{m} |\nabla w^i|^2 = 2 \cdot 2 = 4 \quad \boxed{\text{check}}$$

$\nabla^{\Sigma} w^i = (\nabla^{\mathbb{R}^m} w^i)^{T\Sigma} = (0, \cdots, 1, \cdots, 0)^{T\Sigma}$

$$\Rightarrow 4\,\mathrm{Vol}(\Omega) = \int_{\Omega} \Delta r^2 = 2\int_{\Omega} r \frac{\partial r}{\partial \eta}$$

η: unit outer normal of $\partial\Omega$ wrt Ω



$$\frac{\partial r}{\partial \eta} = \langle \nabla^{\mathbb{R}^3} r, \eta \rangle \leq |\nabla^{\mathbb{R}^3} r| |\eta| = 1$$

$$\Rightarrow \quad 4\, \mathrm{Vol}(\Omega) \leq 2 \int_{\Omega} r \leq 2 \left(\int_{\partial\Omega} r^2 \right)^{\frac{1}{2}} \left(\int_{\partial\Omega} 1 \right)^{\frac{1}{2}}$$

Cauchy-Schwarz

$$= 2 \left(\mathrm{length}(\partial\Omega) \right)^{\frac{1}{2}} \left(\sum_i \int_{\partial\Omega} (w^i)^2 \, ds \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{\pi} \left(\mathrm{length}(\partial\Omega) \right)^{\frac{3}{2}} \Big(\underbrace{\sum_i \int_{\partial\Omega} \left| \frac{\partial w^i}{\partial s} \right|^2}_{\equiv 1} ds \Big)^{\frac{1}{2}}$$

Wirtinger's inequality

$$\left(s = l \frac{\theta}{2\pi} \qquad \int f^2(\theta)\, d\theta \leq \int (f'(\theta))^2 \, d\theta \right.$$

$$\left. \Leftrightarrow \int f^2(s)\, ds \leq \left(\frac{l}{2\pi} \right)^2 \int (f'(s))^2 \, ds \qquad \frac{df}{d\theta} = \frac{df}{ds} \frac{l}{2\pi} \right)$$

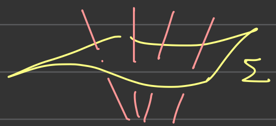
$$\Rightarrow \quad 4\pi\, \mathrm{Vol}(\Omega) \leq \left(\mathrm{length}(\partial\Omega) \right)^2$$

§ III. the argument of Brendle

<u>setting</u> $\Sigma^n \subset \mathbb{R}^{n+m}$, compact, connected, minimal, and $m \geq 2$
with non-empty $\partial\Sigma$

$\begin{cases} \cdot\ \mathbb{R}^N \hookrightarrow \mathbb{R}^{N+1}, \text{ easy to achieve} \\ \cdot\ \text{need this condition later} \end{cases}$

<u>strategy</u> construct a map
$\Phi : N\Sigma \to \mathbb{R}^{n+m}$, which is surject to B^{n+m}
Then, do volume comparison



$N\Sigma \subset \Sigma \times \mathbb{R}^{n+m}$ has dimension $n+m$
$= \{(p, v) \mid p \in \Sigma, v \perp T_p\Sigma\}$

1° The following Neumann problem has a solution

$$\begin{cases} \Delta^\Sigma u = c_0 \\ \langle \nabla^\Sigma u, \eta \rangle = 1 \end{cases}$$

$$c_0 = \frac{Vol(\partial\Sigma)}{Vol(\Sigma)}$$

where c_0 is determined by

$$c_0 \, Vol(\Sigma) = \int_\Sigma \Delta^\Sigma u = \int_{\partial\Sigma} \langle \nabla^\Sigma u, \eta \rangle = Vol(\partial\Sigma)$$

Define $\Phi : T\Sigma \longrightarrow \mathbb{R}^{n+m}$

$$(p, v) \longmapsto \underset{T_p\Sigma}{\underset{\cap}{\nabla^\Sigma u(p)}} + \underset{N_p\Sigma}{\underset{\cap}{v}}$$

2° <u>discussion</u> $\xi \in B^{n+m}$

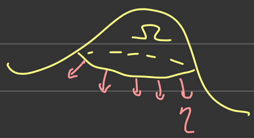
$\Rightarrow |\nabla^\Sigma u(p_0)| = |\xi^T| < 1$

if $\nabla^\Sigma u(p_0) + v_0 = \xi \iff \nabla^\Sigma u(p_0) = \xi^T$

$\iff p_0$ is a critical point of $\Sigma \longrightarrow \mathbb{R}^1$
$p \longmapsto u(p) - \langle p, \xi \rangle$

$\Bigg($ $\gamma(t) \in \Sigma$, $\gamma(0) = p$ $\quad \frac{d}{dt}\left(u(\gamma(t)) - \langle \gamma(t), \xi \rangle \right)$
$= du_{p_0}(\gamma'(0)) - \langle \gamma'(0), \xi \rangle$
$= \langle \nabla u(p_0) - \xi, \gamma'(0) \rangle$ $\quad \gamma'(0)$: any tangent vector $\Bigg)$

Fix $\xi \in B^{n+m}$, consider $f_\xi : p \in \Sigma \longrightarrow u(p) - \langle p, \xi \rangle \in \mathbb{R}^1$



When $p \in \partial\Sigma$, boundary derivative in normal

$$\langle \nabla^\Sigma f_\xi(p), \eta(p) \rangle = \langle \nabla^\Sigma u(p), \eta(p) \rangle - \langle \eta(p), \xi(p) \rangle$$

$$= 1 - \langle \eta, \xi \rangle > 0$$

$$-\langle \eta, \xi \rangle \geq -|\eta||\xi| > -1$$

Hence, $\min_{\Omega} f_\xi$ is achieved in the interior, $\Omega \setminus \partial\Omega$

Denote the point by $p_0 = p_0(\xi)$

$p = (w^1, \dots, w^{n+m})$ $\quad \dfrac{\partial f_\xi}{\partial x^j} = \dfrac{\partial u}{\partial x^j} - \sum_\mu \dfrac{\partial w^\mu}{\partial x^j} \xi^\mu$

vanishes at p_0

$\Downarrow$

$$\frac{\partial u}{\partial x^j} = \langle \xi, \frac{\partial}{\partial x^j} \rangle \Rightarrow \xi^T = \nabla u(p_0)$$

$$v = \xi - \nabla^\Sigma u(p_0) \in N_{p_0}\Sigma$$

$$\frac{\partial^2 f_\xi}{\partial x^i \partial x^j} = \frac{\partial^2 u}{\partial x^i \partial x^j} - \sum_\mu \frac{\partial^2 w^\mu}{\partial x^i \partial x^j} \xi^\mu$$

$$-\Gamma^k_{ij} \frac{\partial f_\xi}{\partial x^k} = -\Gamma^k_{ij} \frac{\partial u}{\partial x^k} + \sum_\mu \Gamma^k_{ij} \frac{\partial w^\mu}{\partial x^k} \xi^\mu$$

$$\Rightarrow \nabla^2 f_\xi = \nabla^2 u - \langle A(-,-), \xi \rangle$$ (semi-) positive definte at p_0

Hessian defined by g

$N\Sigma$-valued, also $\mathbb{R}^{n+m}$-vector

$\Rightarrow \langle A(-,-), \xi \rangle = \langle A(-,-), v \rangle$

3° conclusion of 2° $\quad \mathcal{U} = \{ p \in \Sigma \setminus \partial\Sigma \mid |\nabla^\Sigma u(p)| < 1 \}$

$$\Omega = \{ (p, v) \in N\Sigma \mid p \in \mathcal{U}, \ |\nabla^\Sigma u(p)|^2 + |v|^2 < 1 \}$$

$$\Omega_+ = \{ (p, v) \in \Omega \mid \nabla^2 u - \langle A(-,-), v \rangle \geq 0 \}$$

semi-positive definite on $T_p\Sigma$

$\Rightarrow$ lemma 1 $\quad \Phi(\Omega_+) = B^{n+m}$

differential of $\Phi(p, v) = \nabla u(p) + v = ?$

- let $(x^1, \dots, x^n)$ be geodesic coordinate at p_0, $g_{ij} = \delta_{ij} + O(|x|^2)$
- choose $\{e_1, \dots, e_m\}$ local orthonormal frame for $N\Sigma$ near p_0
- $\rightsquigarrow v = \sum_{k=1}^m y^k e_k$

$$\Phi(x, y) = \underbrace{g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial X}{\partial x^j}}_{T\Sigma} + \underbrace{y^k e_k}_{N\Sigma}$$

$g_{ij} = \delta_{ij}$, $\Gamma_{ij}^k = 0$

@ P_0.
$$\frac{\partial \Phi}{\partial x^i} = \frac{\partial^2 u}{\partial x^i \partial x^j}\frac{\partial X}{\partial x^j} + \frac{\partial u}{\partial x^j} A(\partial_j, \partial_i) + y^k \left(\frac{\partial e_k}{\partial x^i}\right)^T + y^k \left(\frac{\partial e_k}{\partial x^i}\right)^\perp$$

$$\left(\frac{\partial \Phi}{\partial x^i}\right)^T = u_{;ji}\frac{\partial X}{\partial x^j} + y^k \left\langle \frac{\partial e_k}{\partial x^i}, \frac{\partial X}{\partial x^j}\right\rangle \frac{\partial X}{\partial x^j}$$
$$= u_{;ji}\frac{\partial X}{\partial x^j} - \left\langle y_k e_k, A\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\right\rangle \frac{\partial X}{\partial x^j}$$

$$\frac{\partial \Phi}{\partial y^l} = e_l$$

In terms of the basis $\left\{ \frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}, e_1 = \frac{\partial}{\partial y^1}, \cdots, e_m = \frac{\partial}{\partial y^m} \right\}$ at p_0

$$D\Phi = \begin{bmatrix} \nabla^2 u - \langle A(-,-), v\rangle & 0 \\ * & I \end{bmatrix}$$

$\nabla^2 u - \langle A(-,-), v\rangle$ → bilinear form on $T_p\Sigma \longleftrightarrow$ linear map : $T_p\Sigma \circlearrowleft$
by the metric

4° lemma 2 $\quad 0 \le \det(D\Phi) \le \left(\frac{C_0}{n}\right)^n$ on Ω_+

pf: $D\Phi \ge 0$ on Ω_+

$$D\Phi \sim \begin{bmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_n & & \\ & & & 1 & \\ & & & & \ddots \ 1 \end{bmatrix} \quad \lambda_i \ge 0 \qquad \det D\Phi = \lambda_1 \cdots \lambda_n \le \left(\frac{\lambda_1 + \cdots + \lambda_n}{n}\right)^n$$
$$= \left(\frac{tr(D\Phi)}{n}\right)^n$$

$$tr(D\Phi) = \Delta u - \langle \vec{H}, v\rangle = C_0$$ ※

5° $\chi \ge 0$, integrable on B^{n+m}

$$\int_{B^{n+m}} \chi(\xi)\, d\xi_{n+m} \le \int_{\Omega_+} \chi(\Phi(p,v)) \det(D\Phi)\, dv_m\, dp_n$$
$$\le \left(\frac{C_0}{n}\right)^n \int_{p\in\mathcal{U}} \int_{|v|^2 \le 1 - |\nabla u(p)|^2} \chi(\Phi(p,v))\, dv\, dp$$

χ = characteristic function of the annulus $1-\varepsilon \le |\xi| \le 1$

and consider the limit as $\varepsilon \to 1$

$$\text{L.H.S.} = Vol(\partial B^{n+m}) \int_{1-\varepsilon}^{1} r^{n+m-1}\, dr = \frac{Vol(\partial B^{n+m})}{n+m}\left(1 - (1-\varepsilon)^{n+m}\right)$$
$$\frac{Vol(\partial B^{n+m})}{n+m} = Vol(B^{n+m})$$

$$\Rightarrow \lim_{\varepsilon\to 0} \frac{\text{L.H.S}}{\varepsilon} = (n+m)\, Vol(B^{n+m})$$

$$\left(\frac{C_0}{n}\right)^{-n} \text{R.H.S.} = \int_{p\in\mathcal{U}} \int_{|v|^2 \le 1-|\nabla u(p)|^2} \chi(\Phi(p,v))\, dv\, dp$$

$$(1-\varepsilon)^2 \le |\nabla u(p)|^2 + |v|^2 \le 1$$

$$\Rightarrow (1-\varepsilon)^2 - |\nabla u(p)|^2)_+ \le |v|^2 \le 1 - |\nabla u(p)|^2$$

$$\Rightarrow \int_{|v|^2 \le 1-|\nabla u(p)|^2} \chi(\Phi(p,v))\, dv = \mathrm{Vol}(B^m)\left((1-|\nabla u(p)|^2)^{\frac{m}{2}} - ((1-\varepsilon)^2 - |\nabla u(p)|^2)_+^{\frac{m}{2}}\right)$$

Since $m \ge 2$, $b^{\frac{m}{2}} - a^{\frac{m}{2}} \le \frac{m}{2}(b-a)$

$$\le \mathrm{Vol}(B^m)\,\frac{m}{2}\,(2-\varepsilon)\,\varepsilon$$

$$\Rightarrow \text{R.H.S.} \le \left(\frac{C_0}{n}\right)^n \mathrm{Vol}(B^m)\,\frac{m}{2}\,\varepsilon(2-\varepsilon)\,\mathrm{Vol}(\mathcal{U})$$

$$\le \left(\frac{C_0}{n}\right)^n \mathrm{Vol}(B^m)\,\frac{m}{2}\,\varepsilon(2-\varepsilon)\,\mathrm{Vol}(\Sigma)$$

$$\lim_{\varepsilon\to 0} \frac{\text{R.H.S.}}{\varepsilon} \le \left(\frac{C_0}{n}\right)^n \mathrm{Vol}(B^m)\, m\, \mathrm{Vol}(\Sigma)$$

Finally, $(n+m)\,\mathrm{Vol}(B^{n+m}) \le \frac{1}{n^n}\,\frac{\mathrm{Vol}(\partial\Sigma)^n}{\mathrm{Vol}(\Sigma)^{n-1}} \cdot \frac{m}{2}\,\mathrm{Vol}(B^m)$

$$\mathrm{Vol}(\Sigma)^{n-1} \le \frac{1}{n^n}\, m\, \frac{1}{n+m}\,\frac{\mathrm{Vol}(B^m)}{\mathrm{Vol}(B^{n+m})}\,\mathrm{Vol}(\partial\Sigma)^{n-1}$$

When $m=2$, $\frac{1}{n^n}\,\underbrace{\frac{2}{n+2}\,\frac{\overbrace{\mathrm{Vol}(B^2)}^{=\pi}}{\mathrm{Vol}(B^{n+2})}}_{=2\pi\,\mathrm{Vol}(B^n)} = \frac{1}{n^n\,\mathrm{Vol}(B^n)} = \frac{(\mathrm{Vol}(B^n))^{n-1}}{(\mathrm{Vol}(\partial B^n))^n}$

sharp constant ※