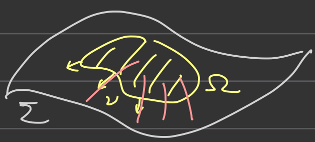
§ I. monotonicity formula

1° boundary term in the divergence formula

recall If V : tangent vector field of compact support

$$\int_\Sigma f \,\mathrm{div}(V) + \langle V, \nabla f \rangle = 0$$



$\begin{cases} \Omega \subset \Sigma \text{ compact.} \\ \text{Assume } \partial\Omega \text{ is (piecewise) smooth} \end{cases}$

Then $\displaystyle\int_\Omega \mathrm{div}(V) = \int_{\partial\Omega} \langle V, \nu \rangle$

where ν is the conormal vector of $\partial\Omega$ (with respect to Ω)

pf: Let θ be the local coordinate for $\partial\Omega$

Then. $(r,\theta) \mapsto \exp_{\theta\in\partial\Omega}(r\nu)$: local coordinate on a neighborhood of $\partial\Omega$

$$g = dr^2 + g_{12}(r,\theta)\, dr\, d\theta + g_{22}(r,\theta)\, d\theta^2$$

$\begin{cases} g_{12}(0,\theta) = 0 & \text{since } \left.\frac{\partial}{\partial r}\right|_{r=0} = \nu \\ g_{11}(r,\theta) \equiv 1, & \text{since } r\text{-curves are geodesics} \end{cases}$

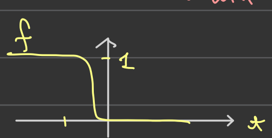
$$\Rightarrow g = \begin{bmatrix} 1 & 0 \\ 0 & g_{22}(0,\theta) \end{bmatrix} + O(r). \quad g^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & g_{22}^{-1}(0,\theta) \end{bmatrix} + O(r)$$

(Orientation on $\partial\Sigma$: if $dr \wedge d\theta > 0$ on Ω

then $\left(\frac{\partial}{\partial r} \lrcorner (dr \wedge d\theta)\right) = d\theta > 0$ on $\partial\Omega$

Contract outward normal gives positive orientation)

↳ will do general Stokes' theorem later



$$f_\varepsilon(r) = f\left(\frac{r}{\varepsilon}\right) = \begin{cases} 1 & \text{if } r < -\varepsilon \\ 0 & \text{if } r > 0 \end{cases} \qquad t = \frac{r}{\varepsilon}$$

$$\int_\Sigma f_\varepsilon \,\mathrm{div}(V) \xrightarrow{\varepsilon \to 0} \int_\Sigma \mathrm{div}(V)$$

$$= -\int_\Sigma \langle V, \nabla f_\varepsilon \rangle$$

Consider series expansion in r

$$df_\varepsilon = f' \frac{1}{\varepsilon} dr \quad \Rightarrow \quad \nabla f_\varepsilon = f' \frac{1}{\varepsilon} \left(\frac{\partial}{\partial r} + O(r) \right)$$

$$V = V_1(0,\theta) \frac{\partial}{\partial r} + V_2(0,\theta) \frac{\partial}{\partial \theta} + O(r)$$

$$\langle V, \nabla f_\varepsilon \rangle \, d\mathrm{vol} = \underline{f' \tfrac{1}{\varepsilon}} \; V_1(0,\theta) \;\; \underline{dr} \left(g_{22}(0,\theta) \, d\theta \right) + f' \frac{1}{\varepsilon} O(r) \, dr \, d\theta$$

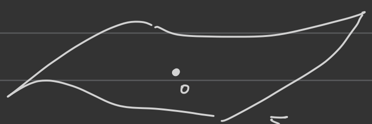
$$\int_{-\infty}^{\infty} f' \frac{1}{\varepsilon} dr = \int_{-\infty}^{\infty} f' \, dt = -1$$

$$\int_{-\infty}^{\infty} f' \frac{r}{\varepsilon} dr = \varepsilon \int_{-\infty}^{\infty} f' t \, dt \xrightarrow{\varepsilon \to 0} 0$$

$$\Rightarrow \; - \int_{\Sigma} \langle V, \nabla f_\varepsilon \rangle \, d\mathrm{vol} \longrightarrow \int_{\partial\Omega} \underbrace{V_1(0,\theta)}_{\substack{\parallel \\ \langle V, \nu \rangle \text{ on } \partial\Omega}} \; \underbrace{g_{22}(0,\theta) \, d\theta}_{d\mathrm{vol} \text{ of } \partial\Omega}$$

※

2° the area of a minimal surface cannot be smaller than that of a 2-plane



Σ

Σ : minimal, choose any point on Σ

By translation, assume that it is the origin

goal study $\mathrm{Vol}(\Sigma \cap B_r)$

Consider the position vector field $W = w^j \frac{\partial}{\partial w^j}$ in $\mathbb{R}^3$

$$W = W^T + W^\perp$$

At any $p \in \Sigma$, choose $\{e_1, e_2\}$: orthonormal basis for $T_p\Sigma$

$$\sum_{i=1}^{2} \langle e_i(W), e_i \rangle = \sum_i \langle e_i(W^T), e_i \rangle + \langle e_i(W^\perp), e_i \rangle$$

$$= \sum_i \langle \nabla_{e_i} W^T, e_i \rangle - \langle W^\perp, (\nabla_{e_i} e_i)^\perp \rangle$$

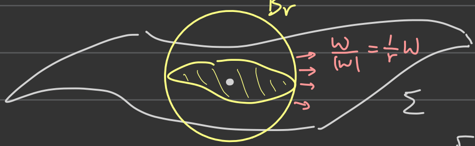
$$= \mathrm{div}(W^T) - \langle W^\perp, \cancelto{0}{H} \rangle$$

$$e_i = e_i^j \frac{\partial}{\partial w^j} \Rightarrow e_i(W) = e_i^j \frac{\partial}{\partial w^j} = e_i \quad \text{the special property of position vector}$$

$$\Rightarrow \quad 2 = \mathrm{div}(W^T) \quad \text{if } \Sigma \text{ is minimal}$$

$\Omega_r = \Sigma \cap B_r$, $\partial\Omega_r = \Sigma \cap \partial B_r$ (suppose Σ is complete)

$$2\,\mathrm{Vol}(\Sigma \cap B_r) = \iint_{\Omega_r} \mathrm{div}(W^T) = \int_{\partial\Omega_r} \langle W^T, \nu \rangle$$



r, $\nabla^{\mathbb{R}^3} r = \frac{1}{r} W$

check $\nabla^{\Sigma} r = \frac{1}{r} W^T$

From the construction, W^T is tangent to Σ and is perpendicular to ∂B_r

$\Rightarrow \nu = W^T / |W^T|$

Hence, $$\int_{\partial\Omega_r} \langle W^T, \nu \rangle = \int_{\partial\Omega_r} |W^T| = \frac{d}{dr} \int_{\Omega_r} |W^T|\,|\nabla^{\Sigma} r| = \frac{d}{dr} \int_{\Omega_r} \frac{|W^T|^2}{r}$$

claim

(In general, $$\int_{\partial\Omega_r} f = \frac{d}{dr} \int_{\Omega_r} f\,|\nabla^{\Sigma} r| \quad \left(|\nabla^{\Sigma} r| = \frac{|W^T|}{r} \right)$$

$$2\,\mathrm{Vol}(\Sigma \cap B_r) = \int_{\partial\Omega_r} |W^T| \quad \left(= r \int_{\partial\Omega_r} \frac{|W^T|}{r} \right)$$

$$= r \frac{d}{dr} \int_{\Omega_r} \frac{|W^T|^2}{r^2} = r \frac{d}{dr} \int_{\Omega_r} \left(1 - \frac{|W^\perp|^2}{r^2} \right)$$

$$= r \frac{d}{dr} \mathrm{Vol}(\Sigma \cap B_r) - r \frac{d}{dr} \int_{\Omega_r} \frac{|W^\perp|^2}{r^2}$$

$$\Rightarrow \frac{d}{dr} \left(\frac{\mathrm{Vol}(\Sigma \cap B_r)}{r^2} \right) = r^{-3} \left(-2\,\mathrm{Vol}(\Sigma \cap B_r) + r\,\mathrm{Vol}(\Sigma \cap \partial B_r) \right)$$

$$= + r^{-2} \frac{d}{dr} \int_{\Omega_r} \frac{|W^\perp|^2}{r^2} = + r^{-2} \int_{\partial\Omega_r} \frac{|W^\perp|^2}{r^2} \frac{1}{|\nabla^{\Sigma} r|} \geq 0$$

$\Rightarrow \dfrac{\mathrm{Vol}(\Sigma \cap B_r)}{r^2}$ is monotone increasing in r

By smoothness of Σ at 0. $\displaystyle \lim_{r \to 0} \frac{\mathrm{Vol}(\Sigma \cap B_r)}{r^2} = \pi$

$\Rightarrow \mathrm{Vol}(\Sigma \cap B_r) \geq \pi r^2$

the monotonicity formula

<u>remark</u> One can study minimal surface from a more measure-theoretical point of view, and consider non-smooth surfaces.

$$\Theta(\Sigma, p) = \lim_{r \to 0} \frac{\mathrm{Vol}(\Sigma \cap B(p,r))}{\pi r^2}$$ detects the smoothness of Σ at p

[about the fact] <u>Co-area formula</u>

$h : \Sigma \to \mathbb{R}$ $\begin{cases} h \geq 0 \\ \text{Lipschitz} \\ \text{proper}, \ \{h(x) \leq a\} \text{ is compact} \end{cases}$

Then, for any integrable f, $\displaystyle\int_{h \leq t} f |\nabla h| = \int_0^t \left(\int_{h^{-1}(t)} f \right) dz$ in (Σ, g)

<u>key</u> If t is a regular value

$$d\mathrm{vol}_\Sigma = dt \wedge d\mathrm{vol}_{h^{-1}(t)} \big/ |\nabla h|$$

§II. second variational formula

$\Sigma \subset \mathbb{R}^3$, ν: normal vector field, f: compact support

$$\Sigma_t = \{ p + t f(p) \nu(p) \mid p \in \Sigma, \ t \in (-\varepsilon, \varepsilon) \}$$

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Vol}(\Sigma_t) = -\int_\Sigma f H \, d\mathrm{vol}$$ (the same as 1st week)

<u>goal</u> Suppose $H = 0$ (critical), compute $\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Vol}(\Sigma_t)$

$$F(x^1, \cdots, x^n, t) = X + t f \nu \qquad g_{ij}(t) = \left\langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \right\rangle$$

<u>recall</u> $\dot{g}_{ij}(0) = -2 f h_{ij}$

$$\frac{d^2}{dt^2} \mathrm{Vol}(F_t) = \frac{d^2}{dt^2} \int \sqrt{\det g_{ij}(t)} \, dx^1 \cdots dx^n$$

$$= \frac{d}{dt} \left(\frac{1}{2} \int g^{kl}(t) \dot{g}_{lk}(t) \sqrt{\det g_{ij}(t)} \, dx^1 \cdots dx^n \right)$$

$$= \int \left(\frac{1}{4} \left(g^{kl}(t) \dot{g}_{lk}(t) \right)^2 + \frac{1}{2} \dot{g}^{kl}(t) \dot{g}_{lk}(t) + \frac{1}{2} g^{kl}(t) \ddot{g}_{lk}(t) \right) d\mathrm{vol}$$

At $t=0$ $\quad g^{kl}\dot{g}_{lk} = -2 g^{kl} f h_{kl} = -2fH = 0$

$$\frac{1}{2}\dot{g}^{kl}(0)\,\dot{g}_{lk}(0) = -\frac{1}{2} g^{ki}\dot{g}_{ij} g^{jl}\dot{g}_{kl}$$
$$= -2f^2 g^{ik} g^{jl} h_{ik} h_{jl}$$
$$= -2|A|^2 f^2$$

$$\ddot{g}_{lk} = \frac{d}{dt}\left(\langle \frac{\partial^2 F}{\partial x^l \partial t}, \frac{\partial F}{\partial x^k}\rangle + (l \leftrightarrow k)\right)$$

$$\frac{\partial^2 F}{\partial t^2} = 0$$

$$= 2\langle \frac{\partial^2 F}{\partial x^l \partial t}, \frac{\partial^2 F}{\partial x^k \partial t}\rangle = 2\langle \frac{\partial (f\nu)}{\partial x^l}, \frac{\partial (f\nu)}{\partial x^k}\rangle$$

$$\frac{\partial (f\nu)}{\partial x^k} = \frac{\partial f}{\partial x^k}\nu - f h_{ki} g^{ip}\frac{\partial X}{\partial x^p}$$

$$= 2 f_l f_k + 2f^2 h_{lj} h_{ki} g^{ji}$$

$$\Rightarrow \frac{1}{2} g^{kl}\ddot{g}_{lk} = |\nabla f|^2 + f^2|A|^2$$

<u>prop</u> $\Sigma^n \subset \mathbb{R}^{n+1}$ minimal, then along $f\nu$

$$\left.\frac{d^2}{dt^2}\right|_{t=0} Vol(\Sigma_t) = \int_\Sigma (|\nabla f|^2 - |A|^2 f^2)\, dvol$$
$$= \int_\Sigma \langle -\Delta f - |A|^2 f, f\rangle\, dvol$$

This called the 2nd variational formula.

$-\Delta - |A|^2$ is the corresponding Jacobi operator (as in the study of variation of geodesics)

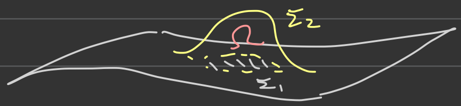
<u>defn</u> a minimal surface Σ is said to be stable if $\int_\Sigma \langle -\Delta f - |A|^2 f, f\rangle\, dvol \geq 0 \quad \forall f$: compact support

§ III. volume minimizer

In general, it is not easy to know that $\left.\frac{d^2}{dt^2}\right|_{t=0} Vol(\Sigma_t) \geq 0$

However, we do know this if Σ is Γ_f for $f: \mathbb{R}^n \to \mathbb{R}^1$

1° <u>tool</u> divergence theorem for a region in $\mathbb{R}^{n+1}$



$V = (V^1, \cdots, V^{n+1})$ vector field on $\mathbb{R}^{n+1}$

$$\iint_\Omega \operatorname{div}(V) = -\int_{\Sigma_1} \langle V, \nu\rangle + \int_{\Sigma_2} \langle V, \nu\rangle$$

ν: upward normal

(for + sign : outer normal)

2° If $f : \mathbb{R}^n \to \mathbb{R}^1$ satisfies $\frac{\partial}{\partial x_i} \frac{\partial_i f}{\sqrt{1+|\nabla f|^2}} = 0$

let $V = \left(\frac{-\partial_1 f}{\sqrt{1+|\nabla f|^2}}, \cdots, \frac{-\partial_n f}{\sqrt{1+|\nabla f|^2}}, \frac{1}{\sqrt{1+|\nabla f|^2}} \right)$ (doesn't depend on x^{n+1})

$\operatorname{div}(V) = 0$, $|V| = 1$

Note that tangent to $\Gamma_f = \operatorname{Span}\{ (0, \cdots, 1, \cdots 0, \partial_i f) \mid i = 1, \cdots, n \}$

$\Rightarrow \nu = V$

Hence, $$\int_{\Gamma_f} \langle V, \nu\rangle = \operatorname{Vol}(\Gamma_f) = \int_{\Sigma_2} \langle V, \nu\rangle \underset{\leq\, |V||\nu| = 1}{\leq} \operatorname{Vol}(\Sigma_2)$$

$\Sigma_2 \leftarrow$ (compactly support) deform of Γ_f

In this case, Γ_f is said to be a volume minimizer.

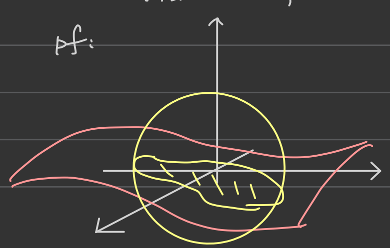
<u>Cor</u> Γ_f is stable.

§IV. Berstein theorem through stability

1° <u>Cor</u> (of volume minimizer) $f : \mathbb{R}^2 \to \mathbb{R}^1$ with Γ_f : minimal

Assume $f(0) = 0$. Then, $\operatorname{Vol}(\Gamma_f \cap B_r) \leq 2\pi r^2$

pf:



$\partial(\Gamma_f \cap B_r)$ divides the sphere, ∂B_r, into two pieces.

Their boundary is also $\partial(\Gamma_f \cap B_r)$

$$\Rightarrow \mathrm{Vol}(\Gamma_f \cap B_r) \le \mathrm{Vol}(\text{any of each piece}) \le \tfrac{1}{2}\mathrm{Vol}(\partial B_r)$$

2° theorem'' $\Sigma \subset \mathbb{R}^3$, complete, oriented minimal surface

$$\begin{cases} \text{which is stable, and} \\ \text{which has Euclidean volume growth, } \mathrm{Vol}(\Sigma \cap B_r) \le cr^2 \quad (\exists c,\ \forall r>0) \end{cases}$$

Then, Σ must be a plane

Γ_f is stable and has Euclidean volume growth

theorem'' $\Rightarrow$ Γ_f is a plane (Bernstein)

pf: i) By stability, $\int_\Sigma |\nabla \chi|^2 - |A|^2\chi^2 \ge 0$ for any $\chi \in C_c^\infty(\Sigma)$

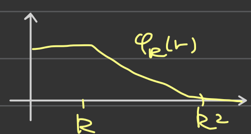
Through limit and approximation, true for Lipschitz χ with compact support

ii) (plug in suitable test function)

$r = \sqrt{x^2+y^2+z^2}$

For any $R > 1$

let $$\varphi_R(r) = \begin{cases} 1 & r \le R \\ 1 - \dfrac{\log(r/R)}{\log R} & R \le r \le R^2 \\ 0 & r \ge R \end{cases}$$



$$\nabla^{\mathbb{R}^3}(\varphi_R(r)) = \varphi_R'(r)\,\nabla^{\mathbb{R}^3} r = -\frac{1}{\log R}\frac{1}{r}\nabla^{\mathbb{R}^3} r$$

unit length

not well-defined at 0

$$\Rightarrow |\nabla^\Sigma \varphi_R| \le \frac{1}{(\log R)^2}\frac{1}{r^2} \quad \begin{cases} \text{for } R \le r \le R^2 \\ \text{vanishes outside this region} \end{cases}$$

iii) $$\int_\Sigma |\nabla^\Sigma \varphi_R|^2 \le \frac{1}{(\log R)^2}\int_{\Sigma \cap (B_{R^2}\setminus B_R)} \frac{1}{r^2} = \frac{1}{(\log R)^2}\sum_{k=1}^{\log R}\int_{\Sigma\cap(B_{e^kR}\setminus B_{e^{k-1}R})}\frac{1}{r^2}$$

consider $R = e^N$

$$\le e^{2-2k}R^{-2}\,\mathrm{Vol}(\Sigma \cap B_{e^kR})$$

$$\le Ce^2 \quad (\text{Euclidea volume growth})$$

iv) $\displaystyle\int_\Sigma |A|^2 \varphi_R^2 = \int_\Sigma |\nabla \varphi_R|^2 \le \frac{Ce^2}{\log R}$

As $R \to \infty$ $\Rightarrow$ $\displaystyle\int_\Sigma |A|^2 \le 0$ $\Rightarrow$ $A \equiv 0$

Hence, ν is a constant vector $\Rightarrow$ Σ is a plane ※

§V. remark on higher dimension (minimal & stable)

1° Simons identity $\quad g^{ij} h_{kl;ij} = -|A|^2 h_{kl}$

$$\Delta A = -|A|^2 A$$

$\Delta A = \mathrm{tr}(\nabla^2 A)$ $\quad$ (ΔA in ∇ notation)

$$\nabla A = dx^i \otimes \nabla_{\partial_i} A$$

$$\nabla^2 A = dx^j \otimes \nabla_{\partial_j}(dx^i \otimes \nabla_{\partial_i} A)$$

$$= dx^j \otimes (dx^i \otimes \nabla_{\partial_j}\nabla_{\partial_i} A + \nabla_{\partial_j} dx^k \otimes \nabla_{\partial_k} A)$$

$$= dx^j \otimes dx^i \otimes (\underbrace{\nabla_{\partial_j}\nabla_{\partial_i} A - \Gamma^k_{ji} \nabla_{\partial_k} A}_{= h_{kl;ij}\, dx^k \otimes dx^l})$$

2° Similarly

$$\Delta |A|^2 = g^{ij} |A|^2_{;ij}$$

$$= g^{ij}\left(\partial_j(\partial_i |A|^2) - (\nabla_{\partial_j}\partial_i)(|A|^2)\right)$$

$\boxed{\text{check}}$ With the previously defined ∇ and inner product on $T^*M \otimes T^*M$, $\quad \partial_j |A|^2 = \langle \nabla_{\partial_j} A, A\rangle + \langle A, \nabla_j A\rangle$

$$\partial_j(\partial_i |A|^2) = 2\,\partial_j(\langle \nabla_{\partial_i} A, A\rangle) = 2\langle \nabla_{\partial_j}\nabla_{\partial_i} A, A\rangle + 2\langle \nabla_{\partial_i} A, \nabla_{\partial_j} A\rangle$$

$$-(\nabla_{\partial_j}\partial_i)(|A|^2) = -2\langle \nabla_{\nabla_{\partial_j}\partial_i} A, A\rangle$$

$$\Rightarrow \Delta |A|^2 = 2\langle \underbrace{g^{ij}(\nabla_{\partial_j}\nabla_{\partial_i} A - \Gamma^k_{ji}\nabla_{\partial_k} A)}_{\Delta A = -|A|^2 A}, A\rangle + 2 g^{ij}\langle \nabla_{\partial_i} A, \nabla_{\partial_j} A\rangle$$

$$\Rightarrow \Delta |A|^2 = -2|A|^4 + 2|\nabla A|^2$$

3° roughly, use $X = |A|$ into the stability condition

$$\int_\Sigma \langle \underbrace{-\Delta X - |A|^2 X, X}\rangle \, dvol \geq 0$$

$\boxed{\text{check}}$

$$-|A|\Delta|A| - |A|^4 = -\frac{1}{2}\Delta|A|^2 + |d|A||^2 - |A|^4$$

$$= \cancel{|A|^4} - |\nabla A|^2 + |d|A||^2 - \cancel{|A|^4}$$

$$\Rightarrow |A|\Delta|A| + |A|^4 = |\nabla A|^2 - |d|A||^2$$

4° Look at $|\nabla A|^2$ and $|d|A||^2$ at some point p.

Choose coordinat such that $g_{ij}|_p = \delta_{ij}$. $h_{ij}|_p = \lambda_i \delta_{ij}$

$$\nabla A = h_{ij;k} \, dx^k \otimes dx^i \otimes dx^j$$

$$\Rightarrow |\nabla A|^2 @ p = \sum_{i,j,k} (h_{ij;k})^2$$

$$d|A| = d(|A|^2)^{\frac{1}{2}}$$

$$= |A|^{-1} \langle \nabla A, A \rangle = |A|^{-1} dx^k \otimes \langle \nabla_k A, A \rangle$$

$$= |A|^{-1} dx^k \otimes \langle h_{ij;k} \, dx^i \otimes dx^j, h_{pq} \, dx^p \otimes dx^q \rangle$$

$$d|A| @ p = |A|^{-1} dx^k \sum_{i,j} h_{ij} h_{ij;k} = dx^k \sum_i \lambda_i h_{ii;k}$$

$$|d|A||^2_{@p} = |A|^{-2} \sum_k \Big(\underline{\sum_i \lambda_i h_{ii;k}}\Big)^2 \rightsquigarrow \text{Cauchy-Schwarz} \quad (k : \text{fixed})$$

$$\leq \Big(\sum_j \lambda_j^2\Big)^{-1} \cdot \sum_k \Big(\Big(\sum_i \lambda_i^2\Big)\Big(\sum_i (h_{ii;k})^2\Big)\Big) = \sum_{i,k} (h_{ii;k})^2$$

$$\Rightarrow \underline{|\nabla A|^2 - |d|A||^2_{@p}} \geq \sum_{i \neq j} (h_{ij;k})^2$$

$$\geq \sum_{i\neq j} (\underbrace{h_{ij;i}}_{k=i})^2 + \sum_{i \neq j} (\underbrace{h_{ij;j}}_{k=j})^2$$

$$= \underline{2 \sum_{i \neq j} (h_{ii;j})^2} \quad \text{by Codazzi}$$

Again $|d|A||^2_{@p} \leq \sum_{i,k} (h_{ii;k})^2 = \sum_{i \neq k} (h_{ii;k})^2 + \sum_i (h_{ii;i})^2$

$$H = 0 \Rightarrow \sum_i h_{ii} = 0 \Rightarrow \sum_i h_{ii;k} = 0$$

$$\Rightarrow -h_{ii;i} = \sum_{i \neq k} h_{ii;k}$$

$\Rightarrow |d|A||^2_{@p} \leq \sum_{i \neq k} (h_{ii;k})^2 + \sum_i \Big(\sum_{k \neq i} h_{ii;k} \Big)^2$ Cauchy-Schwarz (i: fixed)

$$\leq \sum_{i \neq k} (h_{ii;k})^2 + \sum_i \Big((n-1) \sum_{k \neq i} (h_{ii;k})^2 \Big) \leq n \sum_{i \neq k} (h_{ii;k})^2$$

Hence, $|\nabla A|^2 - |d|A||^2 \geq \frac{2}{n} |d|A||^2$ (at any p)

5° $\frac{2}{n} \int_\Sigma |d|A||^2 \leq \int_\Sigma |\nabla A|^2 - |d|A||^2$

$$\leq \int_\Sigma |A| \Delta |A| + |A|^4 \leq 0 \quad \text{by Stability}$$

$\Rightarrow d|A| \equiv 0 \Rightarrow |A| = $ constant

NOT a correct argument : $|A|$ needs not to have compact support

In general, one has to take some cut-off function then play some PDE trick

$\Rightarrow$ Prove theorem" for $\Sigma^{n \leq 5} \subseteq \mathbb{R}^{n+1}$

(Schoen - Simon - Yau 1975)

key plug $|A|$ or $|A|^2$ into stability condition

$\Rightarrow$ obtain useful estimate on A