§I. conformal structure

<u>theorem</u> (Σ^2, g) (oriented). Then, for any $p \in \Sigma$, there exists a coordinate neighborhood such that

$$g = e^{2\lambda}(dx \otimes dx + dy \otimes dy) \qquad \lambda \in C^\infty(\mathcal{U} \subset \mathbb{R}^2; \mathbb{R})$$

This is called the isothermal coordinate

⚠ We are not going to prove this.

c.f. S.-S. Chern. An elementary proof of the existence of isothermal parameters on a surface (1955)

- If Σ is oriented, we may assume $dx \wedge dy > 0$.
- transition ?

$$\begin{cases} g = e^{2\lambda}(dx \otimes dx + dy \otimes dy) = e^{2\tilde{\lambda}}(du \otimes du + dv \otimes dv) \\ \dfrac{\partial(u,v)}{\partial(x,y)} > 0 \end{cases}$$

<u>goal</u> Say more on the transition

$$\begin{aligned}(du)^2 + (dv)^2 &= (u_x dx + u_y dy)^2 + (v_x dx + v_y dy)^2 \\ &= (u_x^2 + v_x^2)\, dx^2 + 2(u_x u_y + v_x v_y)\, dx \cdot dy \\ &\quad + (u_y^2 + v_y^2)\, dy^2 \qquad /\!/ \quad dx^2 + dy^2\end{aligned}$$

$$\Rightarrow \quad u_x^2 + v_x^2 = u_y^2 + v_y^2, \qquad u_x u_y + v_x v_y = 0$$

$$\begin{array}{c} \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \\ \vec{c}_1 \quad \vec{c}_2 \end{array} : \ \det > 0, \quad |c_1| = |c_2|, \quad c_1 \perp c_2$$

Hence, $u_y = -v_x$, $u_x = v_y$

Namely, they obey the Cauchy–Riemann equation

Or equivalently, $u + iv$ is a holomorphic function in $x + iy$ (and vice versa)

$$= \mathrm{Re}\,(dz \otimes d\bar{z})$$
$$= \mathrm{Re}\,((dx + i\,dy) \otimes (dx - i\,dy))$$

- The Laplacian: $g = e^{2\lambda}\left((dx)^2 + (dy)^2\right)$

$$g = \begin{bmatrix} e^{2\lambda} & 0 \\ 0 & e^{2\lambda} \end{bmatrix}, \quad g^{-1} = \begin{bmatrix} e^{-2\lambda} & 0 \\ 0 & e^{-2\lambda} \end{bmatrix} \qquad \sqrt{\det g} = e^{2\lambda}$$

$$\Rightarrow \Delta = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial}{\partial x^j}\right) = e^{-2\lambda}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

Recall $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$

$$\Rightarrow \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial \bar{z}\,\partial z}$$

- $z = x + iy$, $w = u + iv$ transition is (bi-) holomorphic

A surface with such an atlas is called a Riemann surface

Two relevant notions:

i) holomorphic vector field: $V(z)\frac{\partial}{\partial z}$ with $\frac{\partial V}{\partial \bar{z}} = 0$

$$V(z)\frac{\partial}{\partial z} = \tilde{V}(w)\frac{\partial}{\partial w} = \tilde{V}(w(z))\frac{\partial z}{\partial w}\frac{\partial}{\partial z}$$

example On $S^2 = \mathbb{C} \cup \{\infty\}$ $\quad z = w^{-1} \Rightarrow -z^2\frac{\partial}{\partial z} = \frac{\partial}{\partial w}$

ii) holomorphic differential: $\alpha(z)\,dz$ with $\frac{\partial \alpha}{\partial \bar{z}} = 0$

example No such object on S^2, only meromorphic ones.

remark If Σ is closed, Riemann–Roch theory tells us the dimensions of these objects

§II. Weierstrass representation

goal construct minimal surfaces in $\mathbb{R}^3$

1° If $\Sigma \xrightarrow{X} \mathbb{R}^3$ is minimal, components of X are harmonic (with respect to g)

In terms of the isothermal coordinate, $g = e^{2\lambda}(dx^2 + dy^2)$,

$$\Delta X = 0 = e^{-2\lambda}\frac{\partial}{\partial \bar{z}}\left(\frac{\partial X}{\partial z}\right)$$

$\Rightarrow$ (components of) $\frac{\partial X}{\partial z}$ are holomorphic

2° $\frac{\partial X}{\partial w}\frac{\partial w}{\partial z} = \frac{\partial X}{\partial z} \Rightarrow \frac{\partial X}{\partial w} dw = \frac{\partial X}{\partial z} dz$

reason: $X \in \mathcal{C}^\infty(\Sigma; \mathbb{R}^3)$

denote $\frac{\partial X^j}{\partial z}$ by ψ^j $\quad j \in \{1, 2, 3\}$

$$\sum_j (\psi^j)^2 = \sum_j \frac{1}{2}\left(\frac{\partial X^j}{\partial x} - i\frac{\partial X^j}{\partial y}\right)^2$$
$$= \frac{1}{4}\sum_j \left(\left(\left(\frac{\partial X^j}{\partial x}\right)^2 - \left(\frac{\partial X^j}{\partial y}\right)^2\right) - 2i\frac{\partial X^j}{\partial x}\frac{\partial X^j}{\partial y}\right)$$
$$= \frac{1}{4}\left(\underbrace{\left|\frac{\partial X}{\partial x}\right|^2}_{= e^{2\lambda}} - \underbrace{\left|\frac{\partial X}{\partial y}\right|^2}_{= e^{2\lambda}} - 2i\underbrace{\left\langle \frac{\partial X}{\partial x}, \frac{\partial X}{\partial y}\right\rangle}_{= 0}\right) = 0$$

With ψ^j, X can be reconstructed as its complex line integral

3° $(\psi^1)^2 + (\psi^2)^2 + (\psi^3)^2 = 0$

$(\psi^1 + i\psi^2)(\psi^1 - i\psi^2) = -(\psi^3)^2$ $\quad$ transition "cancels"

$$\Rightarrow \begin{cases} h = \dfrac{\psi^3}{\psi^1 - i\psi^2} = \dfrac{-(\psi^1 + i\psi^2)}{\psi^3} & \text{is a meromorphic } \underline{\text{function}} \text{ on } \Sigma \\ 2\varphi\, dz = (\psi^1 - i\psi^2)\, dz & \text{is a holomorphic 1-form} \end{cases}$$

$\psi^3 = \underline{2h\varphi}$, $\quad \psi^1 + i\psi^2 = -h\psi^3 = -2h^2\varphi$

$\Rightarrow \psi^1 = \underline{(1-h^2)\varphi}$, $\quad \psi^2 = \underline{i(1+h^2)\varphi}$

<u>proposition</u> $\Omega \subset \mathbb{C}$ (or some non-compact Riemann surface)

Given h: meromorphic function, $\varphi\, dz$ holomorphic 1-form

$X: \Omega \to \mathbb{R}^3$ defined by

$$\mathrm{Re}\left(\int_{z_0}^{z} (1-h^2)\varphi\, dz,\ \int_{z_0}^{z} i(1+h^2)\varphi\, dz,\ \int_{z_0}^{z} 2h\varphi\, dz\right)$$

is a minimal surface

Pf: Each component is harmonic ※

The conformal factor $e^{2\lambda} = \left|\frac{\partial X}{\partial x}\right|^2 = \left|\frac{\partial X}{\partial y}\right|^2$

$$\Rightarrow e^{2\lambda} = \frac{1}{2}\sum_j \left|\frac{\partial X^j}{\partial x} - i\frac{\partial X^j}{\partial y}\right|^2 = 2\sum_j |\psi^j|^2$$

$$= 2\left(|1-h^2|^2 + |1+h^2|^2 + |2h|^2\right)|\varphi|^2 = 4\,(1+|h|^2)^2|\varphi|^2$$

4° The reason of the choice of h:

$$\psi^j = \frac{1}{2}\left(\frac{\partial X^j}{\partial x} - i\frac{\partial X^j}{\partial y}\right) \Rightarrow \begin{cases} \frac{\partial X}{\partial x} = (\psi^1 + \bar\psi^1,\ \psi^2 + \bar\psi^2,\ \psi^3 + \bar\psi^3) \\ \frac{\partial X}{\partial y} = (i\psi^1 - i\bar\psi^1,\ i\psi^2 - i\bar\psi^2,\ i\psi^3 - i\bar\psi^3) \end{cases}$$

$N : \Sigma \to S^2$ the Gauss map

$$N \mathbin{/\!/_{+}} \frac{\partial X}{\partial x} \times \frac{\partial X}{\partial y} \Rightarrow N \mathbin{/\!/_{+}} \operatorname{Im}(\psi^2\bar\psi^3,\ \psi^3\bar\psi^1,\ \psi^1\bar\psi^2)$$

Re with positive ratio

$$|\varphi|^{-2}\operatorname{Im}\left(i(1+h^2)2\bar h,\ (1-\bar h^2)2h,\ -i(1-h^2)(1+\bar h^2)\right)$$

$$\operatorname{Im}(i(1+h^2)2\bar h) = -\operatorname{Re}((1+h^2)2\bar h) = -\operatorname{Re}(2\bar h + 2h|h|^2)$$

$$= -(\bar h + h|h|^2 + h + \bar h|h|^2) \qquad -(1+|h|^2)\operatorname{Re}(2h)$$

$$= -(1+|h|^2)\operatorname{Re}(2h)$$

$$\operatorname{Im}((1-\bar h^2)2h) = \operatorname{Im}((h - \bar h|h|^2) - (\bar h - h|h|^2))$$

$$= +(1+|h|^2)\operatorname{Im}(2h)$$

$$\operatorname{Im}(-i(1-h^2)(1+\bar h^2)) = \operatorname{Re}(-(1-h^2)(1+\bar h^2)) = \operatorname{Re}(-1 + h^2 - \bar h^2 + |h|^4)$$

$$= -(1+|h|^2)(1-|h|^2)$$

$$\Rightarrow N = \left(\frac{-\operatorname{Re}(2h)}{1+|h|^2},\ \frac{\operatorname{Im}(2h)}{1+|h|^2},\ -\frac{1-|h|^2}{1+|h|^2}\right) \in S^2$$

$\longrightarrow h \in \mathbb{C}$ stereographic projection (of some convention)

<u>upshot</u> The meromorphic function h is equivalent to the Gauss map

<u>recall</u> $X : \Sigma \to \mathbb{R}^3$ minimal.

$\Rightarrow \psi^j = \frac{\partial X^j}{\partial z}$: holomorphic. $\sum_{j=1}^{3} (\psi^j)^2 = 0$

$\Rightarrow h = \psi^3 / (\psi^1 - i\psi^2)$ meromorphic, $\varphi = (\psi^1 - i\psi^2)/2$ holomorphic

$$\Rightarrow X = \mathrm{Re}\left(\int (1-h^2)\varphi\, dz,\ \int i(1+h^2)\varphi\, dz,\ \int 2h\varphi\, dz \right)$$

§ III. two examples

$\Sigma = \mathbb{C} \setminus \{0\}$. $h(z) = z$, $\varphi(z) = \frac{1}{z^2}$

$$\int (1-h^2)\varphi\, dz = \int \left(\frac{1}{z^2} - 1\right) dz = -\frac{1}{z} - z$$

$$\int i(1+h^2)\varphi\, dz = \int \left(\frac{i}{z^2} + i\right) dz = -\frac{i}{z} + iz$$

$$\int 2h\varphi\, dz = \int \frac{2}{z}\, dz = 2\log z$$

$z = e^s e^{i\theta}$

$$-\frac{1}{z} - z = -e^{-s} e^{-i\theta} - e^s e^{i\theta}$$
$$= -2\cosh s \cos\theta - 2i \sinh s \sin\theta$$

$$-\frac{i}{z} + iz = -ie^{-s}e^{-i\theta} + ie^s e^{i\theta}$$
$$= -2\cosh s \sin\theta + 2i \sinh s \cos\theta$$

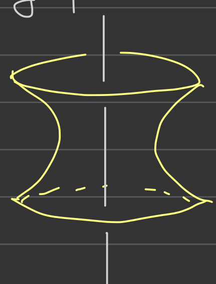
$$2\log z = 2s + 2i\theta$$

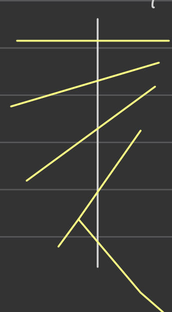
real part $\Rightarrow -2(\cosh s \cos\theta,\ \cosh s \sin\theta,\ s)$ the catenoid

<u>check</u> the only surface of revolution which is minimal.

imaginary part $\Rightarrow -2(\rho \sin\theta,\ -\rho\cos\theta,\ \theta)$: also minimal

the helicoid





Note in general, we can take $e^{i t}\varphi$ $t \in [0, \frac{\pi}{2}]$
($t=0$ catenoid $\longleftrightarrow$ $t=1$ helicoid)
see picture on wikipedia

$\rightsquigarrow$ Deformation between minimal surfaces

Note that $g_t = 4(1+|h|^2)^2 \, |\cancel{e^{it}}\varphi|^2 \, ((dx)^2 + (dy)^2)$.

The deformation is actually isometric

§IV. Bernstein theorem

<u>theorem</u> $f : \mathbb{R}^2 \to \mathbb{R}^1$ If $\Gamma_f \subset \mathbb{R}^3$ is minimal $\Leftrightarrow$ $\frac{\partial}{\partial x_i}\left(\frac{\frac{\partial f}{\partial x_i}}{\sqrt{1+|\nabla f|^2}}\right) = 0$

then f = affine function, $ax+by+c$

$\Leftrightarrow$ Γ_f is a plane

<u>remark</u> higher dimensions $f : \mathbb{R}^n \to \mathbb{R}^1$

- true when $n \le 7$ J. Simons 1968
- false when $n \ge 8$ E. Bombieri, E. De Giorgi, E. Giusti 1969

<u>theorem'</u> (R. Osserman 1961) $\Sigma \hookrightarrow \mathbb{R}^3$. complete and minimal.

If the Gauss map of Σ omits some open subset of S^2

then Σ must be a plane

<u>note</u> (theorem' $\Rightarrow$ theorem) Γ_f : complete $\leftarrow$ check directly

Gauss map of Γ_f : always on the (upper) hemisphere

pf: Consider the universal cover of Σ : $\tilde{\Sigma}$ (at worst immersed)

$$\begin{array}{ccc} \tilde{\Sigma} & & \\ \downarrow & \searrow & \\ \Sigma & \hookrightarrow & \mathbb{R}^3 \end{array}$$

$\tilde{\Sigma}$ is still minimal

$\Rightarrow$ $\tilde{\Sigma}$ cannot be closed $\Rightarrow$ $\tilde{\Sigma} \neq S^2$

Hence, $\tilde{\Sigma}$ is diffeomorphic to $\mathbb{R}^2$ with the complex structure induced by g.

By uniformization theorem (based on Riemann mapping),

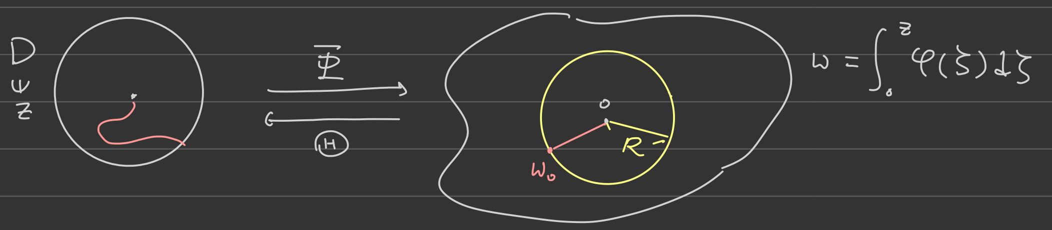
$\tilde{\Sigma}$ must be (biholomorphic to) $\mathbb{C}$ or D

By rotation, we may assume the Gauss map omits a neighborhood of ∞. $\Rightarrow$ Both h and φ are holomorphic and $|h| \le C$, φ is nowhere zero (from metric/immersion condition)

[case 1] If $\tilde{\Sigma} = \mathbb{C}$, h is a bounded entire function
By Liouville, h = constant $\Rightarrow$ Σ is a plane

[case 2] If $\tilde{\Sigma} = D$, consider $\Phi : D \to \mathbb{C}$
$z \mapsto \int_0^z \varphi(\zeta)\, d\zeta$

(metric $= 4(1+|h|^2)^2 |\varphi|^2 |dz|^2 \asymp |\varphi|^2 |dz|^2$
completeness $\Rightarrow \int_\gamma 2(1+|h|^2)|\varphi||dz| = \infty$ if γ goes to ∂D)



$\Phi(0) = 0$. $\Phi'(0) = \varphi(0) \neq 0$
$\Rightarrow$ Φ admits an inverse map, $\textcircled{H}$, on some neighborhood of 0
Suppose that R is the maximum radius such that $\textcircled{H} = \Phi^{-1}$ is defined on $B_R(0)$
$\Rightarrow \exists\, w_0 \in \partial B_R(0)$ such that $\textcircled{H}$ cannot extend to w_0
Consider L = line segment from 0 to w_0
and $C = \textcircled{H}(L)$

<u>claim</u> C must go to ∂D

[If NOT, $C \subset \overline{B_{1-\varepsilon}(0)}$: compact $\Rightarrow \exists\, t_n \to 1$
such that $z_n = \textcircled{H}(t_n w_0) \longrightarrow z_0 \in D$ $\Phi(z_0) = w_0$
Since $\varphi(z_0) \neq 0$, Φ admits a local inverse
$\Rightarrow$ $\textcircled{H}$ extend over w_0]

But then the length of C

$$\infty = |C| \asymp \int_C |\varphi(z)|\,|dz| = \int_L dw < \infty \quad \rightarrow\leftarrow$$

§IV. Simons identity

$\Sigma^n \subset \mathbb{R}^{n+1}$ $\quad (x^1, \cdots, x^n)$ local coordinate for Σ

1° curvature of Σ:

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = R^l{}_{kij}\frac{\partial}{\partial x^l} \qquad \left(R^l{}_{kij} = \frac{\partial}{\partial x^i}\Gamma^l_{kj} - \frac{\partial}{\partial x^j}\Gamma^l_{ki} + \Gamma * \Gamma\right)$$

$$= \left(\nabla_{\partial_i}\nabla_{\partial_j} - \nabla_{\partial_j}\nabla_{\partial_i} - \nabla_{[\partial_i, \partial_j]}\right)\frac{\partial}{\partial x^k}$$

$$\Rightarrow R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\left(V^k \frac{\partial}{\partial x^k}\right) = R^l{}_{kij} V^k \frac{\partial}{\partial x^l} = \left(V^l{}_{;ji} - V^l{}_{;ij}\right)\frac{\partial}{\partial x^l}$$

check (i) For $\alpha_k dx^k$, $\quad \alpha_{k;ji} - \alpha_{k;ij} = -R^l{}_{kij}\alpha_l$

(ii) For $S_{kl}\, dx^k \otimes dx^l$

$$S_{kl;ji} - S_{kl;ij} = -R^p{}_{kij} S_{pl} - R^p{}_{lij} S_{kp}$$

(commuting covariant derivatives must be curvature)

2° Gauss equation $\quad R^l{}_{kij} = g^{lq} h_{iq} h_{jk} - g^{lq} h_{ik} h_{jq}$

Codazzi equation $\quad h_{ki;j} = h_{kj;i}$

3° goal $|A|^2 = g^{ik} g^{jl} h_{ij} h_{kl}$ is a natural defined real-valued function on Σ. $\quad \Delta |A|^2 = ?$

NOT a scalar function

Before that, we can compute "ΔA"

$$\Delta f = \operatorname{tr}\left(X \mapsto \nabla_X\left(g^{ik} f_{;i} \frac{\partial}{\partial x^k}\right)\right) = g^{ij} f_{;ij}$$

$$\frac{\partial}{\partial x^j} \mapsto g^{ik} f_{;ij}\frac{\partial}{\partial x^k}$$

Similarly, define ΔA by $\operatorname{tr}(\nabla^2 A) = g^{ij} h_{kl;ij}\, dx^k \otimes dx^l$

$$\nabla^2 A \in \underbrace{T^*\Sigma \otimes T^*\Sigma}_{\text{take trace here}} \otimes T^*\Sigma \otimes T^*\Sigma$$

4° $\Delta A = ?$ $\quad \left(g^{ij} h_{kl;ij} = ? \right)$

try to switch kl, ij

$h_{kl;i} = h_{ki;l}$ by Codazzi

$$\Rightarrow h_{kl;ij} = h_{ki;lj}$$
$$= h_{ki;jl} - R^p{}_{kjl} h_{pi} - R^p{}_{ijl} h_{kp} \quad \text{by check (ii)}$$

$$\begin{cases} h_{ki;jl} = h_{ik;jl} = h_{ij;kl} & h_{ij} = h_{ji} \text{ and above compuation} \\ R^p{}_{kjl} = g^{pq} h_{jq} h_{kl} - g^{pq} h_{lq} h_{kj} & \\ R^p{}_{ijl} = g^{pq} h_{jq} h_{il} - g^{pq} h_{lq} h_{ij} & \end{cases} \Bigg\} \text{ by Gauss}$$

(only k,l : no sum)

$q\ p \quad j\ i \quad i \quad p \quad j\ q$

$$\Rightarrow g^{ij} h_{kl;ij} = g^{ij} h_{ij;kl}$$
$$- g^{ij} g^{pq} h_{jq} h_{kl} h_{pi} + \cancel{g^{ij} g^{pq} h_{lq} h_{kj} h_{pi}}$$
$$\cancel{- g^{ij} g^{pq} h_{jq} h_{il} h_{kp}} + g^{ij} g^{pq} h_{lq} h_{ij} h_{kp}$$
$$= \underbrace{(g^{ij} h_{ij})}_{= H}{}_{;kl} - |A|^2 h_{kl} + H\, h_{kp} g^{pq} h_{ql}$$

<u>lemma</u> If Σ is minimal, $H = g^{ij} h_{ij} \equiv 0$.

$$\Delta A = -|A|^2 A$$