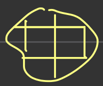
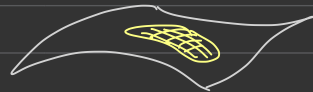
§I. recall geometry of surfaces

<u>setting</u> $\Sigma \subset \mathbb{R}^3$ regular surface



$$\underset{\text{coordinate } (x^1, x^2)}{\mathcal{U} \subset \mathbb{R}^2} \xrightarrow{\ X\ :\ \text{parametrization}\ } \Sigma$$



Σ : oriented. choose a unit normal vector field ν

1° first and second fundamental forms

$$g = \sum_{i,j=1}^{2} g_{ij}\, dx^i \otimes dx^j \qquad \text{where} \quad g_{ij}(x) = \langle \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \rangle$$

usual inner product of $\mathbb{R}^3$

$$A = \sum_{i,j=1}^{2} h_{ij}\, dx^i \otimes dx^j \qquad \text{where} \quad h_{ij}(x) = \langle \frac{\partial^2 X}{\partial x^i \partial x^j}, \nu \rangle$$

In other words, $A(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\, \nu = \left(\frac{\partial^2 X}{\partial x^i \partial x^j}\right)^{\perp}$

⚠ different books have different sign convention

$$= -\langle \frac{\partial X}{\partial x^j}, \frac{\partial \nu}{\partial x^i} \rangle = -\langle \frac{\partial X}{\partial x^i}, \frac{\partial \nu}{\partial x^j} \rangle$$

2° Gauss curvature and mean curvature

g : symmetric and positive definite (inner product)

Denote the inverse of coefficient matrix by

$$g^{ij} = [g^{-1}]_{ij} \qquad \left(\begin{array}{l} \sum_{j=1}^{2} g_{ij} g^{jk} = \delta_i^{\ k} = \sum_{j=1}^{2} g^{kj} g_{ji} \\ \sum_{i,j=1}^{2} g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \text{ is well-defined} \end{array} \right)$$

$$K = \det_g A = (\det g)^{-1} (\det h)$$

$$H = \operatorname{tr}_g A = \sum_{i,j=1}^{2} g^{ij} h_{ji}$$ (some book has a factor of $\frac{1}{2}$)

<u>remark</u>
- (Theorema Egregium) K depends only on g, but not A
- (Gauss–Bonnet) For closed Σ,

$$\iint_{\Sigma} K\, \underbrace{dV}_{\sqrt{\det g}\, dx^1 dx^2} = 2\pi\, \chi(\Sigma)$$

§II. Levi-Civita connection and the Codazzi equation

Through the parametrization, $\frac{\partial}{\partial x^i}$ is identified/visualized as $\frac{\partial X}{\partial x^i}$.

1° Levi-Civita connection

$$\nabla_{\frac{\partial}{\partial x^i}}\left(\sum_{j=1}^{2} a^j \frac{\partial}{\partial x^j}\right) = \left[\sum_{j=1}^{2} \frac{\partial}{\partial x^i}\left(a^j \frac{\partial X}{\partial x^j}\right)\right]^T$$

$$= \sum_{j=1}^{2} \frac{\partial a^j}{\partial x^i}\frac{\partial}{\partial x^j} + \sum_{k=1}^{2} a^k \underbrace{\left(\frac{\partial^2 X}{\partial x^i \partial x^k}\right)^T}_{=\sum_{j=1}^{2} \Gamma^j_{ik}\frac{\partial}{\partial x^j}}$$

$$= \sum_{j=1}^{2} \underbrace{\left(\frac{\partial a^j}{\partial x^i} + \sum_{k=1}^{2} \Gamma^j_{ik} a^k\right)}_{\text{Denote the coefficient by } a^j{}_{;i}} \frac{\partial}{\partial x^j}$$

<u>check</u> $\Gamma^j_{ik} = \frac{1}{2}\sum_{l=1}^{2} g^{jl}\left(\frac{\partial g_{il}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^l}\right)$

2° The dual connection on the cotangent bundle

$V \cong \mathbb{R}^n \Rightarrow V^* \cong \mathbb{R}^n$, but in general NO canonical isomorphism

If V comes with an inner product, $v \longmapsto \eta_v$

$\eta_v(w) = \langle w, v\rangle$

In the current case, $T_p\Sigma \cong T_p^*\Sigma$

$$v = \sum_{i=1}^{2} a^i \frac{\partial}{\partial x^i} \longleftrightarrow \sum_{i,j=1}^{2} (g_{ij}a^i)\, dx^j = v^\flat$$

$$\eta^\# = \sum_{i,j=1}^{2} (g^{ij}\eta_i)\frac{\partial}{\partial x^j} \longleftrightarrow \sum_{i=1}^{2} \eta_i\, dx^i = \eta$$

Define ∇ on $T^*\Sigma$ by

$$\nabla_{\frac{\partial}{\partial x^i}} \eta = (\nabla \eta^\#)^\flat$$

Note that for any $f \in C^\infty(M;\mathbb{R})$, $\nabla(f\eta) = f\nabla\eta + df\cdot\eta$

In terms of component $\nabla_{\frac{\partial}{\partial x^i}}\left(\sum_{j=1}^{2} \eta_j\, dx^j\right) = \sum_{j=1}^{2} \eta_{j;i}\, dx^j$

<u>check</u> $\eta_{j;i} = \frac{\partial \eta_j}{\partial x^i} - \sum_{k=1}^{2} \Gamma^k_{ji}\eta_k$

In other words, $\nabla_{\frac{\partial}{\partial x^j}} dx^i = -\sum_{k=1}^{2} \Gamma^i_{jk}\, dx^k$

⚠ From now on, use the Einstein summation convention

Namely, repeated indices (usually one upper, one lower) are summed.

3° More generally, extend ∇ to sections of $T^*M \otimes T^*M$ by the Leibniz rule:

$$\nabla_{\frac{\partial}{\partial x^i}}(dx^j \otimes dx^k) = (\nabla_{\frac{\partial}{\partial x^i}} dx^j) \otimes dx^k + dx^j \otimes (\nabla_{\frac{\partial}{\partial x^i}} dx^k)$$

$$= (-\Gamma^j_{il}\, dx^l) \otimes dx^k + dx^j \otimes (-\Gamma^k_{il}\, dx^l)$$

$$\Rightarrow \nabla_{\frac{\partial}{\partial x^i}} A = \nabla_{\frac{\partial}{\partial x^i}} (h_{kl}\, dx^k \otimes dx^l) = h_{kl;i}\, dx^k \otimes dx^l$$

where $h_{kl;i} = \frac{\partial h_{kl}}{\partial x^i} - \Gamma^j_{ki} h_{jl} - \Gamma^j_{li} h_{kj}$

$\frac{\partial}{\partial x^i}$ on coefficient, dx^k, dx^l

check $\nabla_{\frac{\partial}{\partial x^i}} g \equiv 0$ (g is "parallel")

4° Decompose the derivative of $\frac{\partial X}{\partial x^j}$ in $\frac{\partial}{\partial x^i}$ into tangent and normal components, and take one more derivative

$$\frac{\partial}{\partial x^k}\left(\frac{\partial^2 X}{\partial x^i \partial x^j} = \Gamma^l_{ij} \frac{\partial X}{\partial x^l} + h_{ij}\, \nu\right)$$

→ tangential component: Gauss equation

→ normal component: Codazzi equation

$\langle \nu, \nu \rangle = 1 \Rightarrow \frac{\partial \nu}{\partial x^k} \perp \nu$

check $\frac{\partial \nu}{\partial x^k} = -h_{kl}\, g^{lj} \frac{\partial X}{\partial x^j}$

$$\left(\frac{\partial^3 X}{\partial x^k \partial x^i \partial x^j}\right)^\perp = \Gamma^l_{ij}\left(\frac{\partial^2 X}{\partial x^k \partial x^l}\right)^\perp + \frac{\partial h_{ij}}{\partial x^k}\, \nu = \left(\frac{\partial h_{ij}}{\partial x^k} + \Gamma^l_{ij} h_{kl}\right) \nu$$

$$= \left(\frac{\partial^3 X}{\partial x^i \partial x^k \partial x^j}\right)^\perp = \left(\frac{\partial h_{kj}}{\partial x^i} + \Gamma^l_{kj} h_{il}\right) \nu$$

$$\Rightarrow \frac{\partial h_{kj}}{\partial x^i} + \Gamma^l_{kj} h_{il} = \frac{\partial h_{ij}}{\partial x^k} + \Gamma^l_{ij} h_{kl}$$

$$h_{ij;k} = \frac{\partial h_{ij}}{\partial x^k} - \Gamma^l_{ki} h_{lj} - \Gamma^l_{kj} h_{il}$$

$$= \left(\frac{\partial h_{kj}}{\partial x^i} + \cancel{\Gamma^l_{kj} h_{il}} - \Gamma^l_{ij} h_{kl}\right) - \Gamma^l_{ki} h_{lj} - \cancel{\Gamma^l_{kj} h_{il}}$$

$$= h_{kj;i} = h_{jk;i}$$

Codazzi equation $h_{ij;k} = h_{jk;i}$

§III. Stokes / divergence theorem

$\dim M = n$ ← let us do the general case

means the chosen orientation

$g = g_{ij}\, dx^i \otimes dx^j$. Σ : oriented. $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n > 0$

$\Rightarrow d\mathrm{vol} = \sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^n$

1° $f \in C^\infty(M;\mathbb{R})$, gradient of f is $\nabla f = (df)^\#$

$\downarrow$

$\nabla f \in \mathfrak{X}(M)$ $\qquad df = \frac{\partial f}{\partial x^i} dx^i \Rightarrow \nabla f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$

2° The (formal) adjoint of $-\nabla$ is the divergence operator

$$\begin{array}{ccc} f & \longmapsto & \nabla f \\ C^\infty(M;\mathbb{R}) & \xrightarrow{\nabla} & \mathfrak{X}(M) \\ C^\infty(M;\mathbb{R}) & \xleftarrow{\mathrm{div}} & \mathfrak{X}(M) \\ \mathrm{div}(V) & \longleftarrow & V \end{array}$$

Define $\mathrm{div}(V)$ by requiring

$$\int_M f\, \mathrm{div}(V)\, d\mathrm{vol} = -\int_M \langle \nabla f, V\rangle\, d\mathrm{vol}$$

$\forall f$ with compact support

div in coordinate? $V = V^i \frac{\partial}{\partial x^i}$

$$\langle \nabla f, V\rangle = \left\langle g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}, V^k \frac{\partial}{\partial x^k}\right\rangle = \underbrace{g_{jk} g^{ij}}_{=\delta^i_k} \frac{\partial f}{\partial x^i} V^k = \frac{\partial f}{\partial x^i} V^i$$

$$-\int \langle \nabla f, V\rangle\, d\mathrm{vol} = -\int \frac{\partial f}{\partial x^i} V^i \sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^n$$

$$= \int f \frac{\partial}{\partial x^i}\left(V^i \sqrt{\det g}\right) dx^1 \wedge \cdots \wedge dx^n = \int f \underbrace{\frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}\left(V^i \sqrt{\det g}\right)}_{=\mathrm{div}(V)} d\mathrm{vol}$$

check i) $\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}(V^i\sqrt{\det g}) = \frac{\partial V^i}{\partial x^i} + \Gamma^i_{ik}V^k =: \mathrm{div}(V)$

ii) $\mathrm{div}(V) = \mathrm{tr}_g(\cdot \mapsto \nabla_\cdot V)$

iii) $(f\,\mathrm{div}(V) + \langle \nabla f, V\rangle)\,d\mathrm{vol} = d(\iota_{(fV)}\,d\mathrm{vol})$

contract an n-form by a vector field

As a Corollary. $\int_M \mathrm{div}(V)\,d\mathrm{vol} = 0$

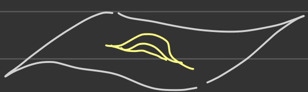
if V has compact support

remark From $\nabla g \equiv 0$,

$$\mathrm{div}(\eta^\#) = \mathrm{div}((\eta_i\,dx^i)^\#) = \mathrm{div}(g^{ij}\eta_i \tfrac{\partial}{\partial x^j})$$
$$= (g^{ij}\eta_i)_{;j} = g^{ij}\eta_{i;j}$$

§ IV. first variational formula

definition' Σ is called a minimal surface if it is a critical state of the area functional



To be more precise, $\forall f \in C^\infty(M;\mathbb{R})$ with compact support.

$\Sigma_t = \{p + t f(p)\nu(p) \mid p \in \Sigma\}$ is a (immersed) surface for $t \in (-\varepsilon, \varepsilon)$

minimal $\Leftrightarrow$ $\left.\frac{d}{dt}\right|_{t=0} \mathrm{Vol}(\Sigma_t) = 0$

$$F(x^1, x^2, t) = X(x^1, x^2) + t f(x^1, x^2)\,\nu(x^1, x^2)$$

$$g_{ij}(t) = \langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\rangle$$

$$\dot g_{ij}(t) = \langle \frac{\partial^2 F}{\partial t\partial x^i}, \frac{\partial F}{\partial x^j}\rangle + (i\leftrightarrow j) = \langle \frac{\partial^2 F}{\partial x^i\partial t}, \frac{\partial F}{\partial x^j}\rangle + (i\leftrightarrow j)$$

$$\dot g_{ij}(0) = \langle \frac{\partial}{\partial x^i}\left(\left.\frac{\partial F}{\partial t}\right|_{t=0}\right), \frac{\partial X}{\partial x^j}\rangle + (i\leftrightarrow j)$$
$$= \langle \frac{\partial}{\partial x^i}(f\nu), \frac{\partial X}{\partial x^j}\rangle + (i\leftrightarrow j)$$
$$= f\langle \frac{\partial \nu}{\partial x^i}, \frac{\partial X}{\partial x^j}\rangle + (i\leftrightarrow j) = -2f\langle \nu, \frac{\partial^2 X}{\partial x^i\partial x^j}\rangle = -2f h_{ij}$$

$$\mathrm{Vol}(\Sigma_t) = \int_\Sigma \sqrt{\det g_{ij}}\, dx^1 \wedge \cdots \wedge dx^n$$

advantage of coordinate computation : $\frac{\partial}{\partial t}$ is clear

<u>check</u> $B(t) : [0,1] \to Gl(n;\mathbb{R})$

$$\frac{d}{dt}\det(B(t)) = \det(B(t))\ \mathrm{tr}\left(B^{-1}(t)\frac{d}{dt}B(t)\right)$$

$t=0$

$$\Rightarrow \left.\frac{d}{dt}\right|_{t=0}\mathrm{Vol}(\Sigma_t) = \int_\Sigma \frac{1}{2} g^{ij}(0)\, \dot{g}_{ij}(0)\, d\mathrm{vol}$$

mean curvature

$$= \int_\Sigma -f\, g^{ij} h_{ij}\, d\mathrm{vol} = \int_\Sigma -f H\, d\mathrm{vol}$$

Hence, $=0\ \forall f \Leftrightarrow H \equiv 0$

This is called the 1^{st} variational formula

<u>definition</u> Σ is called a minimal surface if $H \equiv 0$

§ V. Laplace operator

In $\mathbb{R}^3$ or $\mathbb{C}$, $\Delta = \sum_i \left(\frac{\partial}{\partial x^i}\right)^2$ appears in physics, complex analysis, etc.

1° For $f \in \mathcal{C}^\infty(\Sigma;\mathbb{R})$, consider the Dirichlet energy

$$E(f) = \int_\Sigma |\nabla f|^2\, d\mathrm{vol}$$

critical state ? $\forall \rho \in \mathcal{C}^\infty(\Sigma;\mathbb{R})$ with compact support

$$0 = \left.\frac{d}{dt}\right|_{t=0} E(f+t\rho) = \left.\frac{d}{dt}\right|_{t=0} \int_\Sigma \left(|\nabla f|^2 + 2t\langle\nabla f, \nabla\rho\rangle + t^2|\nabla\rho|^2\right) d\mathrm{vol}$$

$$= 2\int_\Sigma \langle \nabla f, \nabla \rho\rangle\, d\mathrm{vol} = -2\int \rho\, \mathrm{div}(\nabla f)\, d\mathrm{vol}$$

from § III.

$$\Rightarrow \mathrm{div}(\nabla f) \equiv 0$$

<u>definition</u> $\mathrm{div}(\nabla f)$ is called the Laplacian of f, and is denoted by Δf

In local coordinate (from § II),

$$\Delta f = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial f}{\partial x^j}\right)$$

When $g_{ij} = \delta_{ij}$, it is the usual Laplacian as in PDE

2° If Σ is compact without boundary and $\Delta f = 0$,

$$0 = \int f \Delta f \, d\mathrm{vol} = -\int |\nabla f|^2 \, d\mathrm{vol} \Rightarrow \nabla f \equiv 0$$

$\Rightarrow f$ is a constant function

3° $\Sigma \subset \mathbb{R}^3$ (w^1, w^2, w^3) : standard coordinate for $\mathbb{R}^3$

$\rightsquigarrow X(x^1, x^2) = (w^1(x^1,x^2), w^2(x^1,x^2), w^3(x^1,x^2))$

<u>proposition</u> As smooth functions on (Σ, g),

$$\Delta w^i = H \nu^i \leftarrow \text{(i-th component of } \nu)$$

pf: $\Delta w^i = H\nu^i \Leftrightarrow \Delta X = H\nu$ (componentwise act)

$\Leftrightarrow \langle \Delta X, \nu \rangle = H$

$$\langle \Delta X, \nu\rangle = \left\langle \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial X}{\partial x^j}\right), \nu \right\rangle$$

$$= \left\langle g^{ij} \frac{\partial^2 X}{\partial x^i \partial x^j} + \cancel{(\cdots)^j \frac{\partial X}{\partial x^j}}^{\,0}, \nu \right\rangle$$

$$= \left\langle g^{ij}\left(\frac{\partial^2 X}{\partial x^i \partial x^j}\right)^{\perp}, \nu \right\rangle = (\mathrm{tr}_g A) = H$$

※

<u>Cor</u> $\Sigma \subset \mathbb{R}^3$ is a minimal surface if and only if the restriction of standard coordinate functions are harmonic functions (with respect to g)

Cor There is no closed (compact without boundary) minimal surface in $\mathbb{R}^3$.

pf 2) If so, $\Delta w^i = 0 \Rightarrow w^i = \text{constant}$

$\Rightarrow X$ is a constant map $\rightarrow\leftarrow$

pf 1) If $\underset{\text{trace}_g A}{H} = 0 \Rightarrow \underset{\det_g A}{K} \leq 0$ relies on $\dim \Sigma = 2$

But a closed surface in $\mathbb{R}^3$ always has some p where $K_p > 0$ ※