GEOMETRY II: FINAL

MAY 15

(1) [12 points] Consider the *geodesic normal coordinate* at p:

$$(x^1, \dots, x^n) \mapsto \exp_p(\sum_{k=1}^{n} x^k e_k) \qquad (\spadesuit)$$

where $\{e_i\}$ is an orthonormal basis for T_pM. Calculate the series expansion of $g_{ij}(x)$ up to the quadratic term.

Here is the recipe:

(a) $g_{ij}(\mathbf{x}_0) = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$ at $\mathbf{x}_0$. Recall that $\frac{\partial}{\partial x^i}$ mean the tangent of the coordinate curves under the parametrization ($\spadesuit$). That is to say, $\frac{\partial}{\partial x^i}$ at $\mathbf{x}_0$ actually is the tangent vector

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \exp_p\left((\sum_{k=1}^{n} x_0^k e_k) + s e_i \right) .$$

(b) When $\mathbf{x}_0 = \mathbf{0}$, we have seen that $(\exp_p)_*|_\mathbf{0}$ is the identity map, and thus $\frac{\partial}{\partial x^i}|_\mathbf{0} = e_i$. It follows that $g_{ij}(\mathbf{0}) = \delta_{ij}$.

(c) Now, suppose that $\mathbf{x}_0 \neq \mathbf{0}$. Let $\gamma_0(t)$ be the radial geodesic $\exp_p(t(\sum_{k=1}^n x_0^k e_k))$. Parallel transport $\{e_k\}$ along $\gamma_0(t)$. This gives an *orthonormal* basis for $TM|_{\gamma_0(t)}$. Still denote them by $\{e_k\}$.

(d) Try to interpret $\frac{\partial}{\partial x^i}|_{\mathbf{x}_0}$ as a *Jacobi field*. Let

$$\gamma(t, s) = \exp_p\left(t \left[(\sum_{k=1}^{n} x_0^k e_k) + s e_i \right] \right) .$$

Then, $V(t) = \frac{\partial \gamma}{\partial s}|_{s=0}$ is a Jacobi field. A direct calculation shows that $V(0) = 0$ and $V(1) = \frac{\partial}{\partial x^i}|_{\mathbf{x}_0}$. Also, from parallelity, $\frac{\partial \gamma_0}{\partial t} = \sum_{k=1}^n x^k e_k$.

(e) Write $V(t) = \sum_{k=1}^n v^k(t) e_k$. Note that

$$\begin{aligned} f(1) &= f(0) + \int_0^1 f'(t)\,\mathrm{d}t \\ &= f(0) + f'(0) + \int_0^1 (1-t)\, f''(t)\,\mathrm{d}t \\ &= f(0) + f'(0) + \frac{1}{2} f''(0) + \frac{1}{2}\int_0^1 (1-t)^2\, f'''(t)\,\mathrm{d}t \\ &= f(0) + f'(0) + \frac{1}{2} f''(0) + \frac{1}{6} f'''(0) + \frac{1}{6}\int_0^1 (1-t)^3\, f^{(4)}(t)\,\mathrm{d}t \ . \end{aligned}$$

(f) Denote $\frac{\partial \gamma_0}{\partial t} = \sum_{k=1}^{n} x^k e_k$ by T. Recall the Jacobi field equation:

$$\nabla_T \nabla_T V = R(T, V)T \ .$$

It follows that

$$\nabla_T \nabla_T \nabla_T V = (\nabla_T R)((T, V)T) + R(T, \nabla_T V)T$$

where the geodesic equation $\nabla_T T \equiv 0$ has been used.

(2) [4+4+4+8 points] Along a geodesic $\gamma(t)$, a point $q = \gamma(t_0)$ is said to be a *conjugate point* of $p = \gamma(0)$ along γ if there exists a non-trivial Jacobi fields, $J(t)$, along $\gamma|_{[0,t_0]}$ such that $J(0) = 0 = J(t_0)$. A typical example is the north and south poles of a round sphere.

Note that if q is not conjugate to p along $\gamma|_{[0,t_0]}$, any Jacobi field $J(t)$ on it is uniquely determined by $J(0)$ and $J(t_0)$. Thus, we may replace the initial velocity $(\nabla_{\frac{\partial \gamma}{\partial t}} V)|_{t=0}$ by the condition $V(t_0)$.

(a) Write $\gamma(t) = \exp_p(t\mathbf{v})$ for $\mathbf{v} \in T_pM$. Show that q is conjugate to p if and only if q is a singular value of $\exp_p : T_pM \to M$.

(b) Denote $\frac{\partial \gamma}{\partial t}$ by T. For a Jacobi field $J(t)$, show that

$$\langle T, J\rangle = \langle T, J\rangle|_{t=0} + \langle T, \nabla_T J\rangle|_{t=0} \cdot t \ .$$

(c) For two Jacobi fields, J and $\tilde{J}$, prove that

$$\langle \nabla_T J, \tilde{J}\rangle - \langle J, \nabla_T \tilde{J}\rangle \ \text{ is a constant} \ .$$

For a smooth vector field $W(t)$ along a geodesic $\gamma(t)$ ($t \in [0,1]$), define the *index form* by

$$\begin{aligned} I(W,W) &= \int_0^1 \left(|\nabla_T W|^2 + \langle R(W,T)W, T\rangle\right) \mathrm{d}t \\ &= \langle \nabla_T W, W\rangle|_{t=0}^1 - \int_0^1 (\langle \nabla_T \nabla_T W - R(T,W)T, W\rangle)\, \mathrm{d}t \end{aligned}$$

where $T = \frac{\partial \gamma}{\partial t}$. Up to some boundary terms, this is the *second variation* of the energy along W.

(d) Suppose that there are no conjugate points to $p = \gamma(0)$ along $\gamma|_{[0,1]}$. Let W be a smooth vector field on γ with $W(0) = 0$. From the above discussion, there is a unique Jacobi field $J(t)$ on $\gamma|_{[0,1]}$ such that $J(0) = W(0) = 0$ and $J(1) = W(1)$. Prove that

$$I(J,J) \le I(W,W)$$

and the equality holds only when $W \equiv J$.

Hint: Choose a basis $\{E_k\}$ for T_qM where $q = \gamma(1)$. Extend them as a Jacobi field over γ by requiring them to vanish at $p = \gamma(0)$. Denote them by $E_k(t)$. By the no-conjugate points assumption, $\{E_k(t)\}$ is a basis for $T_{\gamma(t)}M$ except at $t = 0$. One can argue that[1]

$$W = \sum_{k=1}^{n} f_k(t) E_k(t)$$

for some smooth functions $f_k(t)$ over $(0, 1)$, where $f_k(0) = 0$ and $\lim_{t\to 0^+}(f_k)'$ exists.

Now,

$$J = \sum_{k=1}^{n} f_k(1) E_k \ ,$$

and

$$I(J, J) = \langle \nabla_T J, J\rangle|_{t=0}^{1} = \sum_{k,\ell=1}^{n} f_k(1) f_\ell(1)\langle E_k'(1), E_\ell(1)\rangle$$

where E_k' means $\nabla_T E_k$. Note that $\nabla_T W = \sum_{k=1}^{n}(f_k' E_k + f_k E_k')$. What can you say about

$$I(W, W) - I(J, J) - \int_0^1 |\sum_{k=1}^{n} f_k' E_k|^2 \, \mathrm{d}t \ ?$$

Remark:

- This can be used to show that after a conjugate point, the geodesic is no longer length minimizing [CE, §1.8].
- Also, with some positivity condition on the curvature, the conjugate points must occur after a certain length, and one can conclude a bound on the diamter of (M, g) [CE, §1.9].

(3) [5+5 points] Given a vector field U, define its *Lie derivative* on a 1-form α by

$$(\mathcal{L}_U \alpha)|_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi_t^*(\alpha) = \lim_{t\to 0} \frac{\varphi_t^*(\alpha|_{\varphi_t(p)}) - \alpha|_p}{t} \qquad (\clubsuit)$$

where φ_t is the one-parameter family of diffeomorphisms generated by U. Note that both terms in the enumerators are elements in T_p^*M, and the subtraction can be performed. This is not a connection; it depends on the behavior of U on an open neighborhood of p.

[1] by the fact that $\nabla_T E_k|_{t=0} \neq 0$ and considering $\frac{1}{t}E_k(t)$

Choose a coordinate chart $\{x^i\}$ on a neighborhood of p. Write $U = u^i(x)\frac{\partial}{\partial x^i}$ The map φ_t locally is given by $\varphi^i(\mathbf{x};t)$ where

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi^i(\mathbf{x};t) = u^i(\varphi(\mathbf{x};t)) \qquad \text{and} \qquad \varphi^i(\mathbf{x};0) = x^i \ .$$

For any 1-form $\alpha = \alpha_i(\mathbf{x})\,\mathrm{d}x^i$,

$$\varphi_t^*\alpha = \alpha_i(\varphi(\mathbf{x};t))\,\frac{\partial\varphi^i(\mathbf{x};t)}{\partial x^j}\,\mathrm{d}x^j \ .$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^*\alpha = \left(\frac{\partial\alpha_i(\mathbf{u})}{\partial u^k}\frac{\partial\varphi^k}{\partial t}\right)\frac{\partial\varphi^i(\mathbf{x};t)}{\partial x^j}\,\mathrm{d}x^j + \alpha_i(\varphi(\mathbf{x};t))\,\frac{\partial^2\varphi^i(\mathbf{x};t)}{\partial t\partial x^j}\,\mathrm{d}x^j$$

where u^k stands for the formal variable $\varphi^k(\mathbf{x};t)$. At $t=0$, $u^k = x^k$. Therefore,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\varphi_t^*\alpha = \frac{\partial\alpha_i}{\partial x^k}u^k\,\mathrm{d}x^i + \alpha_i\frac{\partial u^i}{\partial x^j}\,\mathrm{d}x^j \ .$$

Now, suppose that there is another vector field $V = v^j\frac{\partial}{\partial x^j}$. Recall that

$$[U,V] = u^i\frac{\partial v^j}{\partial x^i}\frac{\partial}{\partial x^j} - v^i\frac{\partial u^j}{\partial x^i}\frac{\partial}{\partial x^j} \ .$$

We compute

$$\begin{aligned}(\mathcal{L}_U\alpha)(V) + \alpha(\mathcal{L}_UV) &= \frac{\partial\alpha_i}{\partial x^k}u^kv^i + \alpha_i\frac{\partial u^i}{\partial x^j}v^j + \alpha_ju^i\frac{\partial v^j}{\partial x^i} - \alpha_jv^i\frac{\partial u^j}{\partial x^i} \\ &= u^k\frac{\partial(\alpha_iv^i)}{\partial u_k} = U(\alpha(V)) \ .\end{aligned}$$

That is to say, it obeys a Leibniz rule. One can similarly use (♣) to define Lie derivative on sections of $\otimes^kT^*M$ as well, and it also obey the Lebniz rule. In particular, for $S \in \Gamma(T^*M \otimes T^*M)$,

$$(\mathcal{L}_US)(V_1,V_2) = U(S(V_1,V_2)) - S(\mathcal{L}_UV_1,V_2) - S(V_1,\mathcal{L}_UV_2) \ .$$

(a) A vector field U is called a *Killing vector field* if it is infinitesimally an isometry, namely,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\varphi_t^*g = 0 \ .$$

Show that U is a Killing vector field if and only if

$$\langle\nabla_{V_1}U,V_2\rangle + \langle\nabla_{V_2}U,V_1\rangle = 0$$

for any two vector fields V_1,V_2.

(b) Suppose that U is a Killing vector field, and γ is a geodesic. Prove that $U|_\gamma$ is a Jacobi field.

Remark: This is intuitively true. U shall generates isometries, which sends geodesics to geodesics. Therefore, it must be a variational field of geodesics of γ.

(4) [5+5+4 points] For a vector field U, define its *divergence* to be the function

$$\mathrm{tr}(V \mapsto \nabla_V U) \ .$$

In terms of coordinate, it reads

$$\mathrm{div}(U) = \frac{\partial U^i}{\partial x^i} + \Gamma^j_{ji} U^i \ .$$

(a) Prove that if the one-parameter family of diffeomorphisms preserves the volume form,

$$\varphi_t^*(\sqrt{\det g_{ij}}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n) = \sqrt{\det g_{ij}}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \quad \text{for any } t \ ,$$

then $\mathrm{div}(U) \equiv 0$.

(b) Show that

$$\mathrm{div}(U)\sqrt{\det g_{ij}}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = \mathrm{d}\left(\iota(U)\sqrt{\det g_{ij}}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n\right) \ .$$

(c) Check that a Killing vector field must be divergence free, $\mathrm{div}(U) \equiv 0$.