GEOMETRY II: HOMEWORK 10

DUE JUNE 12

(1) Let $U = \{(x,y) \in \mathbb{R}^2 \,:\, \sqrt{x^2+y^2} < e^{-1}\}$. Consider the function defined on U by

$$f(x,y) = \log|\log r| \quad \text{when } r = \sqrt{x^2+y^2} \in (0, e^{-1}) \ ,$$

and set $f(0,0) = 0$.

(a) Check that $f \in L^2_1(U)$. Namely,

$$\iint_U \left(|f|^2 + |f_x|^2 + |f_y|^2\right) \mathrm{d}x\mathrm{d}y \quad \text{converges.}$$

(b) Check that $\exp(f) \in L^1$. Namely,

$$\iint_U |\exp(f)| \, \mathrm{d}x\mathrm{d}y \quad \text{converges.}$$

(c) Check that f is not bounded on U, $f \notin L^\infty(U)$.

(2) By using the complex coordinate for the stereographic projection, the standard metric on $\mathbb{S}^2$ is

$$g = \frac{4|\mathrm{d}z|^2}{(1+|z|^2)^2} \ ,$$

where $|\mathrm{d}z|^2 = \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y$. Consider the Möbius transform

$$f(z) = \frac{az+b}{cz+d} \quad \text{where } ad - bc \neq 0 \ .$$

(Note that a, b, c, d take value in $\mathbb{C}$.) It is not hard to check that f defines a diffeomorphism of $\mathbb{S}^2$.

(a) Since f is a diffeomorphism, f^*g is also a Riemannian metric on $\mathbb{S}^2$ with Gaussian curvature 1. Check that f^*g and g are conformal to each other.

(b) Show that one can always find $u \in \mathcal{C}^\infty(\mathbb{S}^2)$ such that $e^{2u}g$ has constant Gaussian curvature 1, with $\sup_{\mathbb{S}^2} |u|$ being arbitrarily large.

(3) In terms of the spherical coordiante, the standard metric on $\mathbb{S}^2$ takes the form

$$g = \mathrm{d}\rho \otimes \mathrm{d}\rho + \sin^2\rho \, \mathrm{d}\theta \otimes \mathrm{d}\theta$$

where $\rho \in [0, \pi]$ and $\theta \in [0, 2\pi]$. When $\rho = 0$ or π, it corresponds to the north or south pole.

(a) Write down the Laplacian, Δ, in terms of this coordinate.

(b) Consider the function $f(\rho) = \log(1 - \cos\rho)$. Calculate Δf.

At first glance, this looks very weird. From the Green's formula/Stokes theorem, $\int \Delta f = 0$ for any $f \in \mathcal{C}^\infty(\mathbb{S}^2)$. The function here is not smooth; it has singularity at $\rho = 0$. In the literature, it is often formulated as

$$\Delta f = c_1\,\delta_N + c_2$$

where δ_N is the delta distribution at the north pole, and c_1, c_2 are constants[1]. It means that

$$\int_{\mathbb{S}^2} f\,\Delta\varphi = c_1\,\varphi(N) + c_2 \int_{\mathbb{S}^2} \varphi$$

for any smooth function φ on $\mathbb{S}^2$. This function f is called the Green's function, and has the similar properties as what you learned in the PDE class.

[1]The constants will make the right hand side has zero integration.