GEOMETRY II: HOMEWORK 9

DUE MAY 29

(1) (a) Suppose that $S_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$ is symmetric, $S_{ij} = S_{ji}$. Check that $S_{ij;k} = S_{ji;k}$. In other words, $\nabla_{\partial_k} S$ is still a symmetric $(0,2)$-tensor.
(b) Suppose that $S_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$ is skew-symmetric, $S_{ij} = -S_{ji}$. Check that $S_{ij;k} = -S_{ji;k}$.

(2) Write $(\nabla_{\partial_k}\nabla_{\partial_\ell} - \nabla_{\partial_\ell}\nabla_{\partial_k} - \nabla_{[\partial_k,\partial_\ell]})\mathrm{d}x^j = \sum_i R_i{}^j{}_{k\,\ell}\mathrm{d}x^i$. Since the musical isomorphism (raising and lowering indices) is parallel, one can find that

$$R_i{}^j{}_{k\,\ell} = g_{ip}\, g^{jq}\, R^p{}_{q\,k\,\ell} = g^{jq} R_{i\,q\,k\,\ell} \ ,$$

where $R^p{}_{q\,k\,\ell}$ is the coefficient of $R(\partial_k, \partial_\ell)\partial_q$ in ∂_p, and $R_{i\,q\,k\,\ell} = \langle R(\partial_k, \partial_\ell)\partial_q, \partial_i\rangle$. Since $R_{i\,q\,k\,\ell} = -R_{q\,i\,k\,\ell}$,

$$R_i{}^j{}_{k\,\ell} = -g^{jq} R_{q\,i\,k\,\ell} = -R^j{}_{i\,k\,\ell} \ .$$

Now, suppose that the Ricci curvature is pointwise proportional to the metric tensor. Namely, there exists a smooth function $f \in \mathcal{C}^\infty(M)$ such that

$$R^\ell{}_{i\,\ell\,j} = f\, g_{ij} \ . \qquad (\star)$$

Since the metric tensor is parallel, taking covariant derivative in ∂_k gives

$$R^\ell{}_{i\,\ell\,j\,;k} = (\partial_k f)\, g_{ij} \ .$$

By the second Bianchi identity,

$$R^\ell{}_{i\,\ell\,j\,;k} + R^\ell{}_{i\,j\,k\,;\ell} + R^\ell{}_{i\,k\,\ell\,;j} = 0 \ .$$

Now, multiply the equation by g^{ij}, and sum over i, j (also ℓ). Note that $\sum_{i,j} g^{ij} g_{ji} = \sum_i \delta^i_i = \dim M$.

(a) Express $\sum_{i,j,\ell} g^{ij} R^\ell{}_{i\,\ell\,j\,;k}$ in terms of $\partial_k f$.

(b) Express $\sum_{i,j,\ell} g^{ij} R^\ell{}_{i\,j\,k\,;\ell}$ in terms of $\partial_k f$.
Hint: By (1) and the above discussion, $g^{ij} R^\ell{}_{i\,j\,k\,;\ell} = g^{\ell m} R_m{}^j{}_{j\,k\,;\ell} = -g^{\ell m} R^j{}_{m\,j\,k\,;\ell}$.

(c) Express $\sum_{i,j,\ell} g^{ij} R^\ell{}_{i\,k\,\ell\,;j}$ in terms of $\partial_k f$.

(d) Combine the above computations to prove that when $\dim M \geq 3$ and connected, $(\star)$ implies that f must be a constant function.

(3) Now, consider the 2-dimensional case. Let $\{e_1, e_2\}$ be an (local) oriented, orthonormal frame for TM, and let $\{\omega^1, \omega^2\}$ be the dual coframe. The metric is $g = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2$. Denote by ω_1^2 the coefficient 1-form of the Levi-Civita connection,

$$\nabla e_1 = \omega_1^2 \otimes e_2 \ , \qquad \mathrm{d}\omega^1 = \omega_1^2 \wedge \omega^2 \ ,$$
$$\nabla e_2 = -\omega_1^2 \otimes e_1 \ , \qquad \mathrm{d}\omega^2 = -\omega_1^2 \wedge \omega^1 \ .$$

The curvature relation in this case is

$$\mathrm{d}\omega_1^2 = -K\,\omega^1 \wedge \omega^2 \quad \text{where } K \text{ is the Gaussian curvature} \ .$$

For a smooth function f, the exterior derivative is $\mathrm{d}f = \sum_i e_i(f)\omega^i$. The gradient vector field is $\nabla f = \sum_i e_i(f)e_i$. The Laplacian of f, Δf, is defined to be the divergence of its gradient vector field, $\mathrm{tr}(X \to \nabla_X(\nabla f))$. In terms of this moving frame notation,

$$\Delta f = \sum_i \left(e_i(e_i(f)) + \sum_j e_i(f)\omega_i^j(e_j) \right) \ .$$

Now, take any smooth function u, and consider the *conformal change* of the metric,

$$\tilde{g} = e^{2u}g = (e^u\omega^1) \otimes (e^u\omega^1) + (e^u\omega^2) \otimes (e^u\omega^2) \ .$$

Calculate the Gaussian curvature of $\tilde{g}$.

Hint: The answer shall involve the original Gaussian curvature K, and the Laplacian of u. Note that $\{e^u\omega^1, e^u\omega^2\}$ is the orthonormal coframe for $\tilde{g}$.