GEOMETRY II: HOMEWORK 8

DUE MAY 22

(1) Let $\mathbb{S}^3$ be the 3-sphere, $\{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 : \sum_{j=1}^4 (x^j)^2 = 1\}$, and let $\bar{g}$ be the metric induced by the standard metric on $\mathbb{R}^4$. Consider the *Clifford torus*,

$$(x^1)^2 + (x^2)^2 = \frac{1}{2} = (x^3)^2 + (x^4)^2 \ .$$

One can parametrize it by

$$\Sigma = \left\{ \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\phi, \sin\phi) \right\} \ . \qquad (\star)$$

(a) Write down the induced metric, g, on Σ by using the parametrization $(\star)$. Then, describe the Levi-Civita connection, ∇, of (Σ, g).

(b) From $(\star)$, $T\Sigma$ has the trivializing sections

$$U = \frac{1}{\sqrt{2}}(-\sin\theta, \cos\theta, 0, 0) \quad \text{and} \quad V = \frac{1}{\sqrt{2}}(0, 0, -\sin\phi, \cos\phi) \ .$$

Choose the following extensions to tangent vector fields on $\mathbb{S}^3$ (in fact, on $\mathbb{R}^4$ as well):

$$\bar{U} = u \cdot (-x^2, x^1, 0, 0) \quad \text{and} \quad \bar{V} = v \cdot (0, 0, -x^4, x^3)$$

where u and v are smooth functions in $((x^1)^2 + (x^2)^2, (x^3)^2 + (x^4)^2)$, and both take value 1 at $(\frac{1}{2}, \frac{1}{2})$. Denote by $\bar{\nabla}$ the Levi-Civita connection of $(\mathbb{S}^3, \bar{g})$. Check directly that $(\bar{\nabla}_U \bar{U})^{T\Sigma} = \nabla_U U$, and also for $\nabla_U V$, $\nabla_V U$ and $\nabla_V V$.

(c) One can write down a unit normal vector field of Σ in $\mathbb{S}^3$:

$$\eta = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, -\cos\phi, -\sin\phi) \ .$$

Since the codimension is 1, $\mathrm{II}(\,\cdot\,,\,\cdot\,) = \langle \mathrm{II}(\,\cdot\,,\,\cdot\,), \eta\rangle\eta$. Find out the second fundamental form (of Σ in $\mathbb{S}^3$).

(d) Justify the Gauss equation in this case directly, $\langle \bar{R}(U, V)V, U\rangle = \cdots$. The curvature of $(\mathbb{S}^3, \bar{g})$ is calculated in the lecture note #11.

(2) Endow the following metric on the unit ball $\{(x^1, \cdots, x^n) \in \mathbb{R}^n : |\mathbf{x}| < 1\}$:

$$g = \frac{4\sum_{j=1}^n \mathrm{d}x^j \otimes \mathrm{d}x^j}{(1 - |\mathbf{x}|^2)^2} \ .$$

Find out its Riemann curvature tensor.

Given $U, V \in T_pM$ which are linearly independent, the *sectional curvature* is defined to be

$$K_p(U,V) = \frac{\langle R(U,V)V, U\rangle}{|U|^2|V|^2 - (\langle U,V\rangle)^2} \ .$$

The denominator is the square of the area of the parallelogram spanned by U, V. For coordinate vector field,

$$K(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \frac{R_{ijij}}{g_{ii}g_{jj} - g_{ij}^2}$$

where

$$R_{ijij} = \langle R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\rangle \ .$$

When M is of two dimension, the sectional curvature depends only on the point p, and is indeed the Gaussian curvature.

Given $U, V \in T_pM$, the *Ricci curvature* is the "partial trace" of the curvature tensor defined by

$$\mathrm{Ric}_p(U,V) = \sum_j \langle R(e_j, U)V, e_j\rangle$$

where $\{e_j\}$ is an orthonormal basis for T_pM. From the properties of Riemann curvature tensor, one can see that Ric is symmetric, i.e. a section of $\mathrm{Sym}(T^*M \otimes T^*M)$. In terms of coordinate notation,

$$\mathrm{Ric} = \sum_j R^j_{k\,j\,\ell}\, \mathrm{d}x^k \otimes \mathrm{d}x^\ell = \sum_{ij} g^{ij}\, R_{i\,k\,j\,\ell}\, \mathrm{d}x^k \otimes \mathrm{d}x^\ell \ .$$

(3) Prove that in three dimensions, the whole Riemann curvature tensor is determined by the Ricci curvature tensor.

Hint: Do this at every point p. We may assume $g_{ij}(p) = \delta_{ij}$, and hence $R^j_{k\,j\,\ell} = R_{j\,k\,j\,\ell}$ at p. The Ricci curvature (at p) has 6 components

$$\mathrm{Ric}_{k\,\ell} = \sum_j R_{j\,k\,j\,\ell}$$

for $k, \ell \in \{1,2,3\}$. You are asked to show $\mathrm{Ric}_{k,\ell}$ completely determines $R_{i\,j\,k\,\ell}$. Remember that the Riemann curvature tensor has some symmetries.

This is no longer true in higher dimensions. To be more precise, there are Ricci flat manifolds which is not flat. The following exercise is one such example.

What is true in general is that the sectional curvatures (curvature on any two plane) determine the whole Riemann curvature tensor. See formula (1.10) in the book by Cheeger and Ebin.

(4) On $\mathbb{S}^3$, consider the following 1-forms

$$\sigma^1 = -x^2 \mathrm{d}x^1 + x^1 \mathrm{d}x^2 - x^4 \mathrm{d}x^3 + x^3 \mathrm{d}x^4 \ ,$$
$$\sigma^2 = -x^3 \mathrm{d}x^1 + x^4 \mathrm{d}x^2 + x^1 \mathrm{d}x^3 - x^2 \mathrm{d}x^4 \ ,$$
$$\sigma^3 = -x^4 \mathrm{d}x^1 - x^3 \mathrm{d}x^2 + x^2 \mathrm{d}x^3 + x^1 \mathrm{d}x^2 \ .$$

By using $\sum_{j=1}^4 (x^j)^2 = 1$, one can show that $\bar{g} = \sum_{k=1}^3 \sigma^k \otimes \sigma^k$ is the metric induced from the standard metric on $\mathbb{R}^4$. The important relation we will need is

$$\mathrm{d}\sigma^i = 2\,\sigma^j \wedge \sigma^k \quad \text{for } (i,j,k) \text{ being cylic permutation of } (1,2,3) \ .$$

On $\{s > 1\} \times \mathbb{S}^3$, consider the following metric:

$$g = \frac{1}{4}\frac{s+1}{s-1} \mathrm{d}s \otimes \mathrm{d}s + 4\frac{s-1}{s+1}\sigma^3 \otimes \sigma^3 + (s^2-1)(\sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2) \ .$$

Show that the Ricci curvature of this metric vanishes identically.

Hint: You may do formal calculation first. Say, $\omega^0 = f(s)\mathrm{d}s$, $\omega^1 = a(s)\sigma^1$, $\omega^2 = b(s)\sigma^2$ and $\omega^3 = c(s)\sigma^3$ form an orthonormal trivializing sections for the contagent bundle. Then, $\mathrm{d}\omega^0 = 0$, and

$$\begin{aligned} \mathrm{d}\omega^1 &= -\omega^1_0 \wedge \omega^0 - \omega^1_2 \wedge \omega^2 - \omega^1_3 \wedge \omega^3 \\ &= a'\,\mathrm{d}s \wedge \sigma^1 + 2a\,\sigma^2 \wedge \sigma^3 \\ &= \frac{a'}{f\,a}\omega^0 \wedge \omega^1 + \frac{2a}{bc}\omega^2 \wedge \omega^3 \ . \end{aligned}$$

Remark: This is called the Euclidean Taub-NUT metric. In fact, one can add a point[1] to it, the manifold can be shown to be diffeomorphic to $\mathbb{R}^4$. From your calculation, you shall find that it gives a Ricci flat, but non-flat metric on $\mathbb{R}^4$.

[1] $\{s = 1\}$, which means the origin in the "polar coordinate"