GEOMETRY II: HOMEWORK 7

DUE APRIL 24

Given $U, V \in T_pM$ which are linearly independent, the *sectional curvature* is defined to be

$$K_p(U,V) = \frac{\langle R(U,V)V, U\rangle}{|U|^2|V|^2 - (\langle U,V\rangle)^2} .$$

The denominator is the square of the area of the parallelogram spanned by U, V.

For coordinate vector field,

$$K(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \frac{R_{ijij}}{g_{ii}g_{jj} - g_{ij}^2} .$$

When M is of two dimension, the sectional curvature depends only on the point p, and is indeed the Gaussian curvature.

(1) Let $\mathbb{H}$ be the upper half plane $\{(x,y) \in \mathbb{R}^2 : y > 0\}$. Endow the Riemannian metric

$$g = \frac{1}{y^2}(\mathrm{d}x^2 + \mathrm{d}y^2) .$$

   (a) Calculate its Christoffel symbols.

   (b) Calculate its Gaussian (sectional) curvature.

   (c) Write (x,y) as $z = x + iy$. For any $(a,b,c,d) \in \mathbb{R}^4$ with $ad - bc = 1$, show that

$$z \mapsto \frac{az+b}{cz+d}$$

   defines an isometry of $(\mathbb{H}, g)$.

   (d) For $p = (0,1)$, find the geodesic with initial velocity $v = (0,1)$.

   (e) Use part (c) and (d) to conclude that $(\mathbb{H}, g)$ is complete.

   (f) Show that the geodesic you find in part (d) minimizes the distance between $(0,1)$ and $(0,s)$ for any $s > 0$.

   Hint: $g \geq \mathrm{d}y^2/y^2$.

   (g) Describe *all* the geodesics of $(\mathbb{H}, g)$ *geometrically* (you do not need to give all the equations).

(2) For any $\alpha > 0$, consider the Riemannian metric

$$g_\alpha = \frac{1}{y^\alpha}(\mathrm{d}x^2 + \mathrm{d}y^2)$$

on $\mathbb{H}$. If $\alpha \neq 2$, prove that $(\mathbb{H}, g_\alpha)$ is not complete.

Hint: Consider the geodesic from $(0,1)$ to the origin or to the infinity.

(3) Show that if $\gamma(t)$ satisfies the geodesic equation, then $|\gamma'(t)|^2$ must be constant.