# GEOMETRY II: HOMEWORK 6

DUE APRIL 17

In the following questions, $(M, g)$ is assumed to be a Riemannian manifold.

(1) Define $\nabla$ by the Koszul formula:

$$\begin{aligned}\langle \nabla_X Y, Z\rangle = \frac{1}{2}\big(&X(\langle Y, Z\rangle) + Y(\langle Z, X\rangle) - Z(\langle X, Y\rangle) \\ &+\langle [X,Y], Z\rangle + \langle [Z,X], Y\rangle - \langle [Y,Z], X\rangle\big)\ .\end{aligned} \tag{K}$$

(a) Note that $[X, fY] = f\cdot[X,Y] + X(f)\cdot Y$. Check that the right hand side of (K) is function linear in $X$ and $Z$.

(b) Check that $\langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = X(\langle Y, Z\rangle)$.

(c) Check that $\langle \nabla_X Y, Z\rangle - \langle \nabla_Y X, Z\rangle = \langle [X,Y], Z\rangle$.

(2) Denote by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ the Riemann curvature tensor. Note that

$$\begin{aligned}(\nabla^3 U)(X,Y,Z) &= (\nabla_X(\nabla^2 U))(Y,Z) \\ &= \nabla_X((\nabla^2 U)(Y,Z)) - (\nabla^2 U)(\nabla_X Y, Z) - (\nabla^2 U)(Y, \nabla_X Z)\ .\end{aligned}$$

(a) Show that $(\nabla^3 U)(X,Y,Z) - (\nabla^3 U)(Y,X,Z) = R(X,Y)\nabla_Z U - \nabla_{R(X,Y)Z}U$.

(b) Show that $(\nabla^3 U)(X,Y,Z) - (\nabla^3 U)(X,Z,Y) = (\nabla_X R)((Y,Z),U) + R(Y,Z)\nabla_X U$.

(c) Use the above two items to prove the second Bianchi identity of the Riemann curvature tensor:

$$(\nabla_X R)((Y,Z),U) + (\nabla_Y R)((Z,X),U) + (\nabla_Z R)((X,Y),U) = 0\ .$$

**Remark**. There is a more systematical/less artificial way to do this, which will be explained later.

(3) Consider the identity endomorphism $\mathbf{I} \in \Gamma(T^*M \otimes TM)$. Since

$$\nabla_X Y = \nabla_X(\mathbf{I}(Y)) = (\nabla_X \mathbf{I})(Y) + \mathbf{I}(\nabla_X Y)\ ,$$

the identity endomorphism is parallel, $\nabla \mathbf{I} \equiv 0$.

In this question, you are asked to check this property by using coordinate calculation. Recall that

$$\nabla \frac{\partial}{\partial x^j} = \sum_{i,k} \Gamma^k_{ji} \, \mathrm{d}x^i \otimes \frac{\partial}{\partial x^k} \ .$$

(a) The identity endomorphism is $\sum_j \mathrm{d}x^j \otimes \frac{\partial}{\partial x^j}$. Check that

$$\sum_j \mathrm{d}x^j \otimes \frac{\partial}{\partial x^j} = \sum_k \mathrm{d}y^k \otimes \frac{\partial}{\partial y^k}$$

for another coordinate system.

(b) Write the induced connection of $T^*M$ in terms of the dual basis as

$$\nabla \mathrm{d}x^j = \sum_{i,k} \Upsilon^j_{ki} \, \mathrm{d}x^i \otimes \mathrm{d}x^k \ .$$

Find out $\Upsilon^j_{ki}$.

(c) Apply item (b) to show that

$$\nabla \left( \sum_j \mathrm{d}x^j \otimes \frac{\partial}{\partial x^j} \right) = 0 \ .$$

(4) Suppose that $M$ is oriented. Choose the coordinate chart such that $\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n$ is the positive orientation.

(a) Check that $\sqrt{\det g_{ij}} \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n = \sqrt{\det \tilde{g}_{ij}} \, \mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \cdots \wedge \mathrm{d}y^n$. Here, $\frac{\partial(y^1, \cdots, y^n)}{\partial(x^1, \cdots, x^n)}$ is always taken to be positive. The $n$-form is called the *volume form* induced by the metric $g$.

(b) Show that the volume form is parallel with respect to the Levi-Civita connection, i.e. $\nabla(\sqrt{\det g_{ij}} \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n) \equiv 0 \ \in \Gamma(T^*M \otimes \Lambda^n T^*M)$.