GEOMETRY II: HOMEWORK 5

DUE MARCH 27

General hint: try to use metric connections.

(1) For a complex vector bundle $E \to M$, the *total Chern class* is defined to be

$$c(E) = \det\left(\mathbf{I} + \frac{i}{2\pi} F_\nabla\right) = 1 + c_1(E) + c_2(E) + \cdots \in \bigoplus_{j\in\mathbb{N}} \mathrm{H}^{2j}_{\mathrm{dR}}(M) \ .$$

Suppose that E, F are two complex vector bundles over M. Prove that

$$c(E \oplus F) = c(E) \wedge c(F) \ .$$

Namely, $c_j(E \oplus F) = \sum_{\ell=0}^{j} c_\ell(E) \wedge c_{j-\ell}(F) \in \mathrm{H}^{2j}_{\mathrm{dR}}(M)$.

(2) For a complex vector bundle $E \to M$, let E^* be its dual bundle. Prove that $c_j(E^*) = (-1)^j c_j(E)$.

For instance, if $c_1(E) \neq 0 \in \mathrm{H}^2_{\mathrm{dR}}(M)$, E and E^* will not be isomorphic.

(3) On a closed, oriented surface Σ, choose a point p, and a coordinate neighborhood of p, $\mathcal{U}$. Assume that $\mathrm{d}x \wedge \mathrm{d}y$ is the (positive) orientation. Let $\mathcal{V} = \Sigma\backslash\{p\}$

For any $n \in \mathbb{Z}$, consider the complex line bundle L_n given by

$$g_{\mathcal{U}\mathcal{V}} = (z/|z|)^n$$

where $z = x + iy$. Evaluate $\int_\Sigma c_1(L_n)$.

(4) For a real vector bundle $E \to M$, choose a connection ∇ and consider

$$\det\left(\mathbf{I} + \frac{s}{2\pi} F_\nabla\right) = 1 + \frac{s}{2\pi}\sigma_1(F_\nabla) + \cdots + \left(\frac{s}{2\pi}\right)^j \sigma_j(F_\nabla) + \cdots \ .$$

For j being odd, show that $\sigma_j(F_\nabla)$ is zero in $\mathrm{H}^{2j}_{\mathrm{dR}}(M)$.

The *Pontryagin class* of E is defined by

$$\det\left(\mathbf{I} + \frac{1}{2\pi} F_\nabla\right) = 1 + p_1(E) + \cdots + p_\ell(E) + \cdots$$

where $p_\ell(E) = \left[\frac{1}{(2\pi)^{2\ell}}\sigma_{2\ell}(F_\nabla)\right] \in \mathrm{H}^{4\ell}_{\mathrm{dR}}(M)$. The first Pontryagin class belongs to $\mathrm{H}^4_{\mathrm{dR}}(M)$. You can see that the first Chern class is basically the trace, and is easier to compute. We will not compute examples of Pontryagin classese in this class.

Another reason is that the Pontryagin classes of a real vector bundle E is equivalent to the Chern classes of $E \otimes_{\mathbb{R}} \mathbb{C}$. In terms of transition functions, it means that $g_{\mathcal{U}\mathcal{V}} : \mathcal{U} \cap \mathcal{V} \to \mathrm{GL}(k; \mathbb{R}) \subset \mathrm{GL}(k; \mathbb{C})$. You can see Proposition 5.38 in [Morita].

(5) You do not need to submit this one. Think about this question: For $T\Sigma$ of a regular surface in $\mathbb{R}^3$, can we consider its Chern class or Pontryagin class? If so, what do we have?