GEOMETRY II: HOMEWORK 4

DUE MARCH 20

(1) Suppose that a rank k vector bundle E carries a bundle metric and metric connection. Compute its curvature by using an orthonormal trivialization. Show that the curvature is a $k \times k$ skew-symmetric 2-forms.

(2) On $\mathcal{U} \times \mathbb{R}^k$, consider $\nabla = \mathrm{d} + A$ and $\tilde{\nabla} = \mathrm{d} + \tilde{A}$, where $\tilde{A} = g^{-1}\mathrm{d}g + g^{-1}Ag$ for a smooth map $g : \mathcal{U} \to \mathrm{GL}(k;\mathbb{R})$. Show that $\tilde{F} = g^{-1}Fg$.

(3) Let Σ be an oriented regular surface in $\mathbb{R}^3$. Denote by ∇ the connection on $T\Sigma$ defined by exterior derivative of the $\mathbb{R}^3$-valued function followed by projection. Choose $p \in \Sigma$, and choose coordinate system such that

- p corresponds to the origin;
- $\frac{\partial X}{\partial u^1}$ and $\frac{\partial X}{\partial u^2}$ are orthonormal at $(0, 0)$;
- $\mathrm{d}u^1 \wedge \mathrm{d}u^2$ is positively oriented.

Hence, the area form at p is $\mathrm{d}u^1 \wedge \mathrm{d}u^2$. Prove that

$$F(\,,\,)\frac{\partial X}{\partial u^1}\bigg|_p = -K(p)\,\mathrm{d}u^1 \wedge \mathrm{d}u^2 \otimes \frac{\partial X}{\partial u^2}$$

where $K(p)$ is the Gaussian curvature of Σ at p.

(4) For the basic example (3) in the note of the first week: a complex line bundle over a closed, oriented surface: construct a connection on that complex line bundle, and evaluate

$$\int\int_\Sigma \frac{i}{2\pi} F\ .$$

Note that the curvature is an $\mathrm{End}(E)$-valued 2-form. For complex line bundles, $\mathrm{End}(E)$ is always a trivial bundle (Why?), and the above integration does make sense.